### 1.21 Pop quiz on Lecture 21 material

1. Let $u = |1+i, 1-i\rangle$ and $v = |i, 1\rangle$ in $\mathbb{C}^2$ with the standard inner product. Determine (with proof) $\|u\|$, $\|v\|$, $\langle u|v\rangle$, $\langle v|u\rangle$, $d(u,v)$, $\cos(\theta(u,v))$ and $\theta(u,v)$.

2. Let $u = 1$ and $v = x$ in $\mathbb{R}[x]_{\le 2}$ with the standard inner product. Determine (with proof) $\|u\|$, $\|v\|$, $\langle u|v\rangle$, $\langle v|u\rangle$, $d(u,v)$, $\cos(\theta(u,v))$ and $\theta(u,v)$.

3. Define a function $\langle,\rangle\colon \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ by
$$\langle (u_1,u_2,u_3),(v_1,v_2,v_3)\rangle = u_1v_1 - u_2v_2 + u_3v_3.$$
Determine (with proof) whether $\langle,\rangle$ is positive definite.

4. Define a $\langle,\rangle\colon \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}$ by
$$\langle (u_1,u_2),(v_1,v_2)\rangle = iu_1\overline{v_1} - iu_2\overline{v_2}.$$
Determine (with proof) whether $\langle,\rangle$ is positive definite.

5. Define a function $\langle,\rangle\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by
$$\langle (u_1,u_2),(v_1,v_2)\rangle = u_1v_1 + 2u_2v_2.$$
Determine (with proof) whether $\langle,\rangle$ is positive definite.

6. Let $V$ be an $\mathbb{R}$-vector space with an innter product.
   (a) Let $u, v \in V$ and suppose that $u$ and $v$ are orthogonal. Prove that
   $$\|u+v\|^2 = \|u\|^2 + |v|^2.$$
   (b) Give an example of $u, v \in \mathbb{R}^2$ such that $u \neq 0$ and $v \neq 0$ and (with the standard inner product on $\mathbb{R}^2$,
   $$\|u+v\|^2 \neq \|u\|^2 + |v|^2.$$

7. Let $A \in M_{n\times n}(\mathbb{C})$ and assume that $A = \overline{A}^T$. Let $\langle,\rangle\colon \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ be given by
$$\langle (u_1,\ldots,u_n),(v_1,\ldots,v_n)\rangle = u^TA\overline{v}.$$
Prove that $\langle,\rangle$ satisfies all the propoerties of an inner product except perhaps the positive definiteness.