Lecture 21: Inner product spaces

Definition (Inner product)

Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Let

$$\begin{array}{lll} \overline{} : \mathbb{C} \to \mathbb{C} & \text{be given by} & \overline{a+bi} = a - bi, \\ \overline{} : \mathbb{R} \to \mathbb{R} & \text{be given by} & \bar{a} = a. \end{array}$$

Let V be an $\mathbb{F}$-vector space. An *inner product on V* is a function

$$\begin{array}{rccc} \langle , \rangle : & V \times V & \to & \mathbb{F} \\ & (u, v) & \mapsto & \langle u, v \rangle \end{array} \quad \text{such that}$$

(1) If $u, v \in V$ then $\langle u, v \rangle = \overline{\langle v, u \rangle}$,

(2) If $u, v \in V$ and $\alpha \in \mathbb{F}$ then $\langle \alpha u, v \rangle = \alpha \langle u, v \rangle$,

(3) if $u, v, w \in V$ then $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$,

(4) (Positive semi-definite) If $u \in V$ then $\langle u, u \rangle \in \mathbb{R}_{\geqslant 0}$.

(5) (Positive definite) If $u \in V$ and $\langle u, u \rangle = 0$ then $u = 0$.

Definition (Length, distance, angles.)

Let V be an $\mathbb{F}$-vector space with an inner product.
Length is the function $\| \ \|\colon V \to \mathbb{R}_{\geqslant 0}$ given by

$$\|u\|^2 = \langle u, u \rangle.$$

Distance is the function $d\colon V \times V \to \mathbb{R}_{\geqslant 0}$ given by

$$d(u, v) = \|v - u\|.$$

Angle is the function $\theta\colon V \times V \to \mathbb{R}_{[0,\pi]}$ given by

$$\cos(\theta(u, v)) = \frac{\mathrm{Re}(\langle u, v \rangle)}{\|u\| \cdot \|v\|}.$$

Vecotrs u, v are *orthogonal* if $\langle u, v \rangle = 0$.

Definition (Standard inner products.)

(1) $$\langle , \rangle : \begin{array}{ccc} \mathbb{R}^n \times \mathbb{R}^n & \to & \mathbb{R} \\ (u, v) & \mapsto & \langle u | v \rangle \end{array} \quad \text{given by}$$

$$\langle u_1, u_2, \ldots, u_n | v_1, v_2, \ldots, v_n \rangle = u_1 v_1 + \cdots + u_n v_n,$$

(2) $$\langle , \rangle : \begin{array}{ccc} \mathbb{C}^n \times \mathbb{C}^n & \to & \mathbb{C} \\ (u, v) & \mapsto & \langle u | v \rangle \end{array} \quad \text{given by}$$

$$\langle u_1, u_2, \ldots, u_n | v_1, v_2, \ldots, v_n \rangle = u_1 \overline{v_1} + \cdots + u_n \overline{v_n},$$

(3) $\mathbb{F}[x]_{\leqslant n} \times \mathbb{F}[x]_{\leqslant n} \to \mathbb{F}$ given by

$$\begin{aligned} &\langle a_0 + a_1 x + \cdots + a_n x^n \mid c_1 + c_1 x + \cdots + c_n x^n \rangle \\ &\qquad = \int_0^1 (a_0 + a_1 x + \cdots a_n x^n)(\bar{c}_0 + \bar{c}_1 x + \cdots + \bar{c}_n x^n)\, dx. \end{aligned}$$

Example IP5. Let $u = |1+i, 1 = i\rangle$ and $v = |i, 1\rangle$ in $\mathbb{C}^2$ with the standard inner product. Then

$$\begin{aligned}
\langle u|v\rangle &= \langle 1+i, 1-i|i, 1\rangle = (1+i)\bar{i} + (1-i)\bar{1} \\
&= (1-i)(-i) + 1 - i = -i + 1 + 1 - i = 2 - 2i, \\
\langle v|u\rangle &= \langle i, 1|1+i, 1-i\rangle = i\overline{(1+i)} + 1 \cdot \overline{(1-i)} \\
&= i(1-i) + 1 + i = i + 1 + 1 + i = 2 + 2i, \\
\langle u|u\rangle &= \langle 1+i, 1-i|1+i, 1-i\rangle = (1+i)\overline{(1+i)} + (1-i)\overline{(1-i)} \\
&= (1+i)(1-i) + (1-i)(1+i) = 1 + 1 + 1 + 1 = 4, \\
\langle u|u\rangle &= \langle i, 1 \mid i, 1\rangle = i \cdot \bar{i} + 1 \cdot \bar{1} = i(-i) + 1 = 1 + 1 = 2, \\
d(u,v) &= \sqrt{\langle 1, -i|1, -i\rangle} = \sqrt{1 \cdot \bar{1} + (-i)\overline{(-i)}} = \sqrt{1+1} = \sqrt{2}, \\
\cos(\theta(u,v)) &= \frac{\mathrm{Re}(\langle u|v\rangle)}{\|u\| \cdot \|v\|} = \frac{\mathrm{Re}(2-2i)}{2\sqrt{2}} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}},
\end{aligned}$$

So $\theta(u,v) = \frac{\pi}{4}$.

Example IP7. Let $u = 1$ and $v = x$ in $\mathbb{R}[x]_{\leqslant 2}$ with the standard inner product. Then

$$\begin{aligned}
\langle u|u\rangle &= \langle 1|1\rangle = \int_0^1 \, dx = x\Big]_0^1 = 1, \\
\langle v|v\rangle &= \langle x|x\rangle = \int_0^1 x^2 \, dx = \tfrac{1}{3}x^3\Big]_0^1 = \tfrac{1}{3} - 0 = \tfrac{1}{3}, \\
\langle u|v\rangle &= \langle 1|x\rangle = \int_0^1 x \, dx = \tfrac{1}{2}x^2\Big]_0^1 = \tfrac{1}{2} - 0 = \tfrac{1}{2}, \\
d(u, v) &= \sqrt{\langle x-1|x-1\rangle} = \sqrt{\int_0^1 (x-1)^2 \, dx} \\
&= \sqrt{\tfrac{1}{3}(x-1)^3\Big]_0^1} = \sqrt{0 - \tfrac{1}{3}(-1)^3} = \sqrt{\tfrac{1}{3}} = \tfrac{1}{\sqrt{3}}, \\
\cos(\theta(u, v)) &= \frac{\mathrm{Re}(\langle u|v\rangle)}{\|u\| \cdot \|v\|} = \frac{\frac{1}{2}}{1 \cdot \frac{1}{\sqrt{3}}} = \frac{\sqrt{3}}{2}.
\end{aligned}$$

So $\theta(u, v) = \frac{\pi}{6}$.

Example IP2. The map $\langle , \rangle \colon \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ given by

$$\begin{aligned}\langle (u_1, u_2, u_3), (v_1, v_2, v_3) \rangle &= u_1 v_1 - u_2 v_2 + u_3 v_3 \\ &= \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}\end{aligned}$$

has

$$\langle (0,1,0), (0,1,0) \rangle = 0 - 1 + 0 = -1 \notin \mathbb{R}_{\geqslant 0}.$$

So $\langle , \rangle$ is not positive definite.

Example IP6. The map $\langle , \rangle \colon \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{R}$ given by

$$\langle (u_1, u_2), (v_1, v_2) \rangle = i u_1 \overline{v_1} - i u_2 \overline{v_2} = \begin{pmatrix} \overline{v_1} & \overline{v_2} \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

has

$$\langle (1,0), (1,0) \rangle = i \cdot 1 \cdot \bar{1} - i \cdot 0 \cdot \bar{0} = i \notin \mathbb{R}_{\geqslant 0}.$$

So $\langle , \rangle$ is not positive definite.

Example IP3. The map $\langle , \rangle \colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ given by

$$\begin{aligned}
\langle (u_1, u_2), (v_1, v_2) \rangle &= 2u_1 v_1 - 2u_1 v_2 - 2u_2 v_1 + u_2 v_2 \\
&= \begin{pmatrix} v_1 & v_2 \end{pmatrix} \begin{pmatrix} 2 & -2 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}
\end{aligned}$$

has

$$\begin{aligned}
\langle (u_1, u_2), (u_1, u_2) \rangle &= 2u_1^2 - 2u_2 u_2 - 2u_2 u_1 + 3u_2^2 \\
&= 2u_1^2 - 4u_1 u_2 + 3u_2^2 \\
&= 2(u_1^2 - 2u_1 u_2 + u_2^2) + u_2^2 \\
&= (u_1 - u_2)^2 + u_2^2 \in \mathbb{R}_{\geqslant 0}.
\end{aligned}$$

Assume $\langle (u_1, u_2), (u_1, u_2) \rangle = 0$.
Then $2(u_1 - u_2)^2 + u_2^0 = 0$ then $u_2^2 = 0$ and $(u_1 - u_2)^2 = 0$.
So $u_2 = 0$ and $u_1 = u_2 = 0$. So $(u_1, u_2) = 0$.

So $\langle , \rangle$ is positive definite.

Example IP1. The map $\langle , \rangle \colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ given by

$$\langle (u_1, u_2), (v_1, v_2) \rangle = u_1 v_1 + 2u_2 v_2 = \begin{pmatrix} v_1 & v_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

has $\langle (u_1, u_2), (u_1, u_2) \rangle = u_1^2 + 2u_2^2 \in \mathbb{R}_{\geqslant 0}$.

Assume $\langle (u_1, u_2), (u_1, u_2) \rangle = 0$.
Then $u_1^2 + 2u_2^0 = 0$ so that $u_1^2 = 0$ and $2u_2^2 = 0$.
So $(u_1, u_2) = 0$.

So $\langle , \rangle$ is positive definite.

Example IP8. Let V be an $\mathbb{F}$-vector space with an inner product. Let $u, v \in V$ and suppose that u and v are orthogonal. Then

$$\begin{aligned} \|u + v\|^2 &= \langle u + v, u + v \rangle = \langle u, u + v \rangle + \langle v, u + v \rangle \\ &= \langle u, u \rangle + \langle u, v \rangle + \langle v, u \rangle + \langle v, v \rangle \\ &= \|u\|^2 + 0 + 0 + \|v\|^2 \\ &= \|u\|^2 + \|v\|^2. \end{aligned}$$

This is the Pythagorean theorem.

Example IPA1. Let $A \in M_{n\times n}(\mathbb{C})$ satisifying $A = \bar{A}^t$ and let $\langle , \rangle \colon \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ be given by

$$\langle (u_1, \ldots, u_n), (v_1, \ldots, v_n) \rangle = (u_1, \ldots, u_n) A \begin{pmatrix} \overline{v_1} \\ \vdots \\ \overline{v_n} \end{pmatrix} = u^t A \bar{v},$$

If $u, v \in C^n$ then

$$\begin{aligned} \overline{\langle v, u \rangle} &= \overline{(v^t A \bar{u})} = \overline{(\bar{u}^t A^t v)}^t = (u^t \bar{A}^t \bar{v})^t \\ &= (u^t A \bar{v})^t = \langle u, v \rangle, \end{aligned}$$

and if $\alpha \in \mathbb{C}$ and $u, v \in \mathbb{C}^n$ then

$$\langle \alpha u, v \rangle = (\alpha u)^t A \bar{v} = \alpha u^t A v = \alpha \langle u, v \rangle$$

and, if $u, v, w \in \mathbb{C}^n$ then

$$\begin{aligned} \langle u + v, w \rangle &= (u + v)^t A \bar{w} = (u^t + v^t) A \bar{w} \\ &= u^t A \bar{w} + v^t A \bar{w} = \langle u, w \rangle + \langle v, w \rangle. \end{aligned}$$

So $\langle , \rangle$ satisfies all the properties of an inner product, except perhaps the positive definiteness.