Lecture 20: Symmetric, Hermitian, unitary and orthogonal matrices

Definition (Transpose of a matrix)

Let $A \in M_{t\times s}(\mathbb{Q})$. The *transpose of A* is $A^T \in M_{s\times t}(\mathbb{Q})$ given by

$$(A^T)_{ij} = A_{ji}, \quad \text{for } i \in \{1,\dots,s\} \text{ and } j \in \{1,\dots,t\}.$$

Example M4. If $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ then $A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$.

Definition (Symmetric, Hermitian, Unitary, Orthogonal matrices.)

A *symmetric matrix* is $A \in M_{n\times n}(\mathbb{C})$ such that $A = A^T$.

An *orthogonal matrix* is $A \in M_{n\times n}(\mathbb{C})$ such that $AA^T = 1$.

A *Hermitian matrix* is $A \in M_{n\times n}(\mathbb{C})$ such that $A = \overline{A}^T$.

A *unitary matrix* is $A \in M_{n\times n}(\mathbb{C})$ such that $A\overline{A}^T = 1$.

Example IP22. Let $A \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}$. Since

$$A^* = \bar{A}^T = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} = A \quad \text{and} \quad B^* = \bar{B}^T = \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix} \neq B$$

then A is Hermitian and B is not Hermitian.

Example IP21. The matrix $U = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}$ is unitary since

$$UU^* = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Example IP15. $Q = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is orthogonal since

$$\begin{aligned} QQ^T &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Definition (The general linear group)

The *general linear group* $GL_n(\mathbb{R})$ is the set

$$GL_n(\mathbb{R}) = \left\{ A \in M_{n\times n}(\mathbb{R}) \;\middle|\; \begin{array}{l} \text{there exists } A^{-1} \in M_{n\times n}(\mathbb{R}) \\ \text{such that } AA^{-1} = 1 \text{ and } A^{-1}A = 1 \end{array} \right\}$$

Definition (The orthogonal and unitary groups.)

The *orthogonal group* $O_n(\mathbb{R})$ is the set

$$O_n(\mathbb{R}) = \{A \in M_{n\times n}(\mathbb{R}) \mid AA^T = 1\}.$$

The *unitary group* $U_n(\mathbb{C})$ is the set

$$U_n(\mathbb{C}) = \{A \in M_{n\times n}(\mathbb{C}) \mid A\overline{A}^T = 1\}.$$

Example IP17. Assume $Q \in O_n(\mathbb{R})$. Then $1 = QQ^T$ and

$1 = \det(1) = \det(QQ^T) = \det(Q)\det(Q^T) = \det(Q)\det(Q) = \det(Q)^2.$

So $\det(Q) \in \{1, -1\}$.

Definition (Standard inner products on $\mathbb{R}^n$ and $\mathbb{C}^n$)

(a) The *standard inner product on $\mathbb{R}^n$* is $\langle , \rangle \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + \cdots + x_n y_n,$$

if $\mathbf{x} = |x_1, \ldots, x_n\rangle$ and $\mathbf{y} = |y_1, \ldots, y_n\rangle$.

(b) The *standard inner product on $\mathbb{C}^n$* is $\langle , \rangle \colon \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$ given by

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 \overline{y_1} + \cdots + x_n \overline{y_n},$$

if $\mathbf{x} = |x_1, \ldots, x_n\rangle$ and $\mathbf{y} = |y_1, \ldots, y_n\rangle$.

Example IP16. Let $u, v \in \mathbb{R}^n$ and let $Q \in O_n(\mathbb{R})$. Then

$$\langle u | v \rangle = u^T v \quad \text{and}$$

$$\langle Qu | Qv \rangle = (Qu)^T Qv = u^T Q^T Qv = u^T \cdot 1 \cdot v = u^T v.$$

So $\langle Qu | Qv \rangle = \langle u | v \rangle$.

Definition (Orthonormal basis of $\mathbb{R}^n$ and of $\mathbb{C}^n$)

A *basis of $\mathbb{R}^n$* is a subset $\{b_1, \ldots, b_n\}$ of $\mathbb{R}^n$ such that

every vector in $\mathbb{R}^n$ is a unique $\mathbb{R}$-linear combination of $b_1, \ldots, b_n$.

A *basis of $\mathbb{C}^n$* is a subset $\{b_1, \ldots, b_n\}$ of $\mathbb{C}^n$ such that

every vector in $\mathbb{C}^n$ is a unique $\mathbb{C}$-linear combination of $b_1, \ldots, b_n$.

An *orthonormal basis of $\mathbb{R}^n$* is a basis of $\{b_1, \ldots, b_n\}$ of $\mathbb{R}^n$ such that

$$\text{if } i, j \in \{1, \ldots, n\} \quad \text{then} \quad \langle b_i, b_j \rangle = \delta_{ij},$$

where $\langle , \rangle$ is the standard inner product on $\mathbb{R}^n$.

An *orthonormal basis of $\mathbb{C}^n$* is a basis of $\{b_1, \ldots, b_n\}$ of $\mathbb{C}^n$ such that

$$\text{if } i, j \in \{1, \ldots, n\} \quad \text{then} \quad \langle b_i, b_j \rangle = \delta_{ij},$$

where $\langle , \rangle$ is the standard inner product on $\mathbb{C}^n$.

Theorem

Let $A \in M_{n\times n}(\mathbb{R})$. *Then* $A \in GL_n(\mathbb{R})$ *if and only if the columns of* A *form a basis of* $\mathbb{R}^n$.

Theorem (Diagonalization)

Let $A \in M_{n\times n}(\mathbb{F})$. *The matrix* A *has* n *linearly independent eigenvectors* $\mathbf{p}_1, \ldots, \mathbf{p}_n \in \mathbb{F}^n$ *with eigenvalues* $\lambda_1, \ldots, \lambda_n$ *if and only if* $A = PDP^{-1}$ *where,*

$$P = \begin{pmatrix} | & & | \\ \mathbf{p}_1 & \cdots & \mathbf{p}_n \\ | & & | \end{pmatrix} \quad \text{and} \quad D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n) = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

so that $\mathbf{p}_1, \ldots, \mathbf{p}_n$ *are the columns of* P *and* D *is the diagonal matrix with diagonal entries* $\lambda_1, \ldots, \lambda_n$.

Theorem

Let $A \in M_{n\times n}(\mathbb{C})$. *Then* $A \in U_n(\mathbb{C})$ *if and only if the columns of* A *form an orthonormal basis of* $\mathbb{C}^n$ *with respect to the standard inner product on* $\mathbb{C}^n$.

Theorem (Hermitian diagonalization)

Let $A \in M_{n\times n}(\mathbb{C})$ *be a Hermitian matrix. If* $\mathbf{p}_1, \dots, \mathbf{p}_n \in \mathbb{C}^n$ *are orthonormal eigenvectors for* A *with eigenvalues* $\lambda_1, \dots, \lambda_n$ *and*

$$P = \begin{pmatrix} | & & | \\ \mathbf{p}_1 & \cdots & \mathbf{p}_n \\ | & & | \end{pmatrix} \quad \text{and} \quad D = \operatorname{diag}(\lambda_1, \dots, \lambda_n) = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

then P *is unitary and* $A = PD\overline{P}^T$.

Theorem

Let $A \in M_{n\times n}(\mathbb{R})$*. Then* $A \in O_n(\mathbb{R})$ *if and only if the columns of* A *form an orthonormal basis of* $\mathbb{R}^n$ *with respect to the standard inner product on* $\mathbb{R}^n$*.*

Theorem (Real symmetric diagonalization)

Let $A \in M_{n\times n}(\mathbb{R})$ *be a symmetric matrix. If* $\mathbf{p}_1, \dots, \mathbf{p}_n \in \mathbb{R}^n$ *are orthonormal eigenvectors for* A *with eigenvalues* $\lambda_1, \dots, \lambda_n$ *and*

$$P = \begin{pmatrix} | & & | \\ \mathbf{p}_1 & \cdots & \mathbf{p}_n \\ | & & | \end{pmatrix} \quad \text{and} \quad D = \operatorname{diag}(\lambda_1, \dots, \lambda_n) = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

then P *is orthogonal and* $A = PD\overline{P}^T$*.*

Example IP18. The characteristic polynomial of the symmetric matrix

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad \text{is} \quad \begin{aligned} \det(A - t) &= (1-t)^2 - 1 \\ &= 1 - 2t + t^2 - 1 = t^2 - 2t \\ &= (t-0)(t-2). \end{aligned}$$

Then

$$\tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

are eigenvectors of length 1 with eigenvalues 0 and 2, respectively. Then

$$Q = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{is orthogonal}$$

and

$$A = Q \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} Q^T.$$

Example IP23. The characteristic polynomial of the Hermitian matrix

$$A = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \quad \text{is} \quad \begin{aligned} \det(A - t) &= (1 - t)^2 - (-i) \cdot i \\ &= 1 - 2t + t^2 - 1 = t^2 - 2t \\ &= (t - 0)(t - 2). \end{aligned}$$

Then

$$\tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \quad \text{and} \quad \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

are eigenvectors of A of length 1 with eigenvalues 0 and 2, respectively. Then

$$U = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \quad \text{is unitary}$$

and

$$A = U \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \bar{U}^T.$$