

3 Some Proofs

3.1 Some proofs: Matrices and operations

3.1.1 $M_{m \times n}(\mathbb{F})$ is an $\mathbb{F}$ -module

Proposition 3.1. *Let $m, n \in \mathbb{Z}_{>0}$ and let $M_{m \times n}(\mathbb{F})$ be the set of $m \times n$ matrices with entries in $\mathbb{F}$.*

- (a) *If $A, B, C \in M_{m \times n}(\mathbb{F})$ then $A + (B + C) = (A + B) + C$.*
- (b) *If $A, B \in M_{m \times n}(\mathbb{F})$ then $A + B = B + A$.*
- (c) *If $A \in M_{m \times n}(\mathbb{F})$ then $0 + A = A$ and $A + 0 = A$.*
- (d) *If $A \in M_{m \times n}(\mathbb{F})$ then $(-A) + A = 0$ and $A + (-A) = 0$.*
- (e) *If $A \in M_{m \times n}(\mathbb{F})$ and $c_1, c_2 \in \mathbb{F}$ then $c_1 \cdot (c_2 \cdot A) = (c_1 c_2) \cdot A$.*
- (f) *If $A \in M_{m \times n}(\mathbb{F})$ and $1 \in \mathbb{F}$ is the identity in $\mathbb{F}$ then $1 \cdot A = A$.*

Proof.

- (a) To show: $(A + B) + C = A + (B + C)$.

To show: If $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$ then $((A + B) + C)(i, j) = (A + (B + C))(i, j)$.

Assume $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$.

To show: $((A + B) + C)(i, j) = (A + (B + C))(i, j)$.

$$\begin{aligned} ((A + B) + C)(i, j) &= (A + B)(i, j) + C(i, j) = (A(i, j) + B(i, j)) + C(i, j) \\ &= A(i, j) + (B(i, j) + C(i, j)), \quad \text{since addition is associative in } \mathbb{F}, \\ &= A(i, j) + (B + C)(i, j) = (A + (B + C))(i, j). \end{aligned}$$

- (b) To show: $A + B = B + A$.

To show: If $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$ then $(A + B)(i, j) = (B + A)(i, j)$.

Assume $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$.

To show: $(A + B)(i, j) = (B + A)(i, j)$.

$$\begin{aligned} (A + B)(i, j) &= A(i, j) + B(i, j) \\ &= B(i, j) + A(i, j), \quad \text{since } \mathbb{F} \text{ has commutative addition,} \\ &= (B + A)(i, j). \end{aligned}$$

- (c) To show: (ca) $0 + A = A$.

(cb) $A + 0 = A$.

(ca) To show: If $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$ then $(0 + A)(i, j) = A(i, j)$.

Assume $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$.

To show: $(0 + A)(i, j) = A(i, j)$.

$$(0 + A)(i, j) = 0(i, j) + A(i, j) = 0 + A(i, j) = A(i, j).$$

(cb) To show: If $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$ then $(A + 0)(i, j) = A(i, j)$.

Assume $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$.

To show: $(A + 0)(i, j) = A(i, j)$.

$$(A + 0)(i, j) = A(i, j) + 0(i, j) = A(i, j) + 0 = A(i, j).$$

(d) To show: (da) $A + (-A) = 0$.

(db) $(-A) + A = 0$.

(da) To show: If $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$ then $(A + (-A))(i, j) = 0(i, j)$.

Assume $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$.

To show: $(A + (-A))(i, j) = 0(i, j)$.

$$\begin{aligned} (A + (-A))(i, j) &= A(i, j) + (-A)(i, j) = A(i, j) + (-A(i, j)) \\ &= 0 = 0(i, j). \end{aligned}$$

(db) To show: If $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$ then $((-A) + A)(i, j) = 0(i, j)$.

Assume $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$.

To show: $((-A) + A)(i, j) = 0(i, j)$.

$$\begin{aligned} ((-A) + A)(i, j) &= (-A)(i, j) + A(i, j) = (-A(i, j)) + A(i, j) \\ &= 0 = 0(i, j). \end{aligned}$$

(e) Assume $A \in M_{m \times n}(\mathbb{F})$ and $c_1, c_2 \in \mathbb{F}$.

To show $c_1 \cdot (c_2 \cdot A) = (c_1 c_2) \cdot A$.

To show: If $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$ then $(c_1 \cdot (c_2 \cdot A))(i, j) = ((c_1 c_2) \cdot A)(i, j)$.

Assume $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$.

To show: $(c_1 \cdot (c_2 \cdot A))(i, j) = ((c_1 c_2) \cdot A)(i, j)$.

$$\begin{aligned} (c_1 \cdot (c_2 \cdot A))(i, j) &= c_1 \cdot ((c_2 \cdot A)(i, j)) = c_1 \cdot (c_2 \cdot A(i, j)) \\ &= (c_1 \cdot c_2) \cdot A(i, j) = ((c_1 \cdot c_2) \cdot A)(i, j) \end{aligned}$$

(f) Assume $A \in M_{m \times n}(\mathbb{F})$ and $1 \in \mathbb{F}$.

To show: (fa) $1 \cdot A = A$,

(fb) $A \cdot 1 = A$.

(fa) To show: If $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$ then $(1 \cdot A)(i, j) = A(i, j)$.

Assume $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$.

To show: $(1 \cdot A)(i, j) = A(i, j)$.

$$(1 \cdot A)(i, j) = 1 \cdots A(i, j) = A(i, j), \quad \text{since } 1 \text{ is the identity in } \mathbb{F}.$$

(fb) To show: If $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$ then $(A \cdot 1)(i, j) = A(i, j)$.

Assume $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$.

To show: $(A \cdot 1)(i, j) = A(i, j)$.

$$(A \cdot 1)(i, j) = A(i, j) \cdot 1 = A(i, j), \quad \text{since } 1 \text{ is the identity in } \mathbb{F}.$$

□

3.1.2 $M_n(\mathbb{F})$ is an $\mathbb{F}$ -algebra

Proposition 3.2. Let $n \in \mathbb{Z}_{>0}$ and let $M_n(\mathbb{F})$ be the set of $n \times n$ matrices in $\mathbb{F}$.

- (a) If $A, B, C \in M_n(\mathbb{F})$ then $A + (B + C) = (A + B) + C$.
- (b) If $A, B \in M_n(\mathbb{F})$ then $A + B = B + A$.
- (c) If $A \in M_n(\mathbb{F})$ then $0 + A = A$ and $A + 0 = A$.
- (d) If $A \in M_n(\mathbb{F})$ then $(-A) + A = 0$ and $A + (-A) = 0$.
- (e) If $A, B, C \in M_n(\mathbb{F})$ then $A(BC) = (AB)C$.
- (f) If $A, B, C \in M_n(\mathbb{F})$ then $(A + B)C = AC + BC$ and $C(A + B) = CA + CB$.
- (g) If $A \in M_n(\mathbb{F})$ then $1A = A$ and $A1 = A$.

Proof. Parts (a), (b) (c) and (d) are the cases $m = n$ of Proposition 1.1

- (e) To show: $(AB)C = A(BC)$.

To show: If $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$ then $((AB)C)(i, j) = (A(BC))(i, j)$.

Assume $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$.

To show: $((AB)C)(i, j) = (A(BC))(i, j)$.

$$\begin{aligned}
 ((AB)C)(i, j) &= \sum_{k=1}^n (AB)(i, k)C(k, j) \\
 &= \sum_{k=1}^n \sum_{l=1}^n (A(i, l)B(l, k))C(k, j) \\
 &= \sum_{k=1}^n \sum_{l=1}^n A(i, l)(B(l, k)C(k, j)) \\
 &= \sum_{l=1}^n A(i, l)(BC)(l, j) \\
 &= (A(BC))(i, j).
 \end{aligned}$$

- (f) To show: (fa) $(A + B)C = AC + BC$.

(fb) $C(A + B) = CA + CB$.

- (fa) To show: If $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$ then $((A + B)C)(i, j) = (AC + BC)(i, j)$.

Assume $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$.

To show: $((A + B)C)(i, j) = (AC + BC)(i, j)$.

$$\begin{aligned}
 ((A + B)C)(i, j) &= \sum_{k=1}^n (A + B)(i, k)C(k, j) = \sum_{k=1}^n (A(i, k) + B(i, k))C(k, j) \\
 &= \sum_{k=1}^n A(i, k)C(k, j) + B(i, k)C(k, j), \\
 &= \sum_{k=1}^n A(i, k)C(k, j) + \sum_{k=1}^n B(i, k)C(k, j), \\
 &= (AC)(i, j) + (BC)(i, j) = ((AC) + (BC))(i, j) \\
 &= (AC + BC)(i, j).
 \end{aligned}$$

where the third equality follows from the distributive property in $\mathbb{F}$.

- (fb) To show: If $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$ then $(C(A + B))(i, j) = (CA + CB)(i, j)$.
 Assume $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$.
 To show: $(C(A + B))(i, j) = (CA + CB)(i, j)$.

$$\begin{aligned}
 (C(A + B))(i, j) &= \sum_{k=1}^n C(i, k)(A + B)(k, j) = \sum_{k=1}^n C(i, k)(A(k, j) + B(k, j)) \\
 &= \sum_{k=1}^n C(i, k)A(k, j) + C(i, k)B(k, j), \\
 &= \sum_{k=1}^n C(i, k)A(k, j) + \sum_{k=1}^n C(i, k)B(k, j), \\
 &= (CA)(i, j) + (CB)(i, j) = ((CA) + (CB))(i, j) \\
 &= (CA + CB)(i, j).
 \end{aligned}$$

where the third equality follows from the distributive property in $\mathbb{F}$.

- (g) To show: (ga) $1 \cdot A = A$,
 (gb) $A \cdot 1 = A$.

- (ga) To show: If $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$ then $(1 \cdot A)(i, j) = A(i, j)$.
 Assume $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$.
 To show: $(1 \cdot A)(i, j) = A(i, j)$.

$$\begin{aligned}
 (1 \cdot A)(i, j) &= \sum_{k=1}^n 1(i, k)A(k, j) = 1(i, i)A(i, j) + \sum_{\substack{k \in \{1, \dots, n\} \\ k \neq i}} 1(i, k)A(k, j) \\
 &= 1 \cdot A(i, j) + 0 = A(i, j).
 \end{aligned}$$

- (gb) To show: If $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$ then $(A \cdot 1)(i, j) = A(i, j)$.
 Assume $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$.
 To show: $(A \cdot 1)(i, j) = A(i, j)$.

$$\begin{aligned}
 (A \cdot 1)(i, j) &= \sum_{k=1}^n A(i, k)1(k, j) = A(i, j)1(j, j) + \sum_{\substack{k \in \{1, \dots, n\} \\ k \neq j}} A(i, k)1(k, j) \\
 &= 0 + A(i, j) \cdot 1 = A(i, j).
 \end{aligned}$$

□

3.1.3 Properties of transpose

Proposition 3.3. Let $m, n \in \mathbb{Z}_{>0}$, let $M_{m \times n}(\mathbb{F})$ be the set of $m \times n$ matrices with entries in $\mathbb{F}$, and let $M_n(\mathbb{F})$ be the set of $n \times n$ matrices in $\mathbb{F}$.

- (a) If $A, B \in M_{m \times n}(\mathbb{F})$ then $(A + B)^t = A^t + B^t$,
 (b) If $A \in M_{m \times n}(\mathbb{F})$ and $c \in \mathbb{F}$ then $(c \cdot A)^t = c \cdot A^t$,
 (c) If $A, B \in M_n(\mathbb{F})$ then $(AB)^t = B^t A^t$.
 (d) If $A \in M_n(\mathbb{F})$ then $(A^t)^t = A$.

Proof.

(a) Assume $A \in M_{m \times n}(\mathbb{F})$.

To show $(A + B)^t = A^t + B^t$.

To show: If $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$ then $(A + B)^t(i, j) = (A^t + B^t)(i, j)$.

Assume $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$.

To show: $(A + B)^t(i, j) = (A^t + B^t)(i, j)$.

$$\begin{aligned} (A + B)^t(i, j) &= (A + B)(j, i) = A(j, i) + B(j, i) \\ &= A^t(i, j) + B^t(i, j) = (A + B)^t(i, j). \end{aligned}$$

(b) Assume $A \in M_{m \times n}(\mathbb{F})$ and $c \in \mathbb{F}$.

To show $(c \cdot A)^t = c \cdot A^t$.

To show: If $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$ then $((c \cdot A)^t)(i, j) = (c \cdot A^t)(i, j)$.

Assume $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$.

To show: $((c \cdot A)^t)(i, j) = (c \cdot A^t)(i, j)$.

$$((c \cdot A)^t)(i, j) = (c \cdot A)(j, i) = c \cdot A(j, i) = c \cdot A^t(i, j) = (c \cdot A^t)(i, j)$$

(c) Assume $A, B \in M_n(\mathbb{F})$.

To show: $(AB)^t = B^t A^t$.

To show: If $i, j \in \{1, \dots, n\}$ then $(AB)^t(i, j) = (B^t A^t)(i, j)$.

Assume $i, j \in \{1, \dots, n\}$.

To show: $(AB)^t(i, j) = (B^t A^t)(i, j)$.

$$\begin{aligned} (AB)^t(i, j) &= (AB)(j, i) = \sum_{k=1}^n A(j, k)B(k, i) \\ &= \sum_{k=1}^n A^t(k, j)B^t(i, k) = \sum_{k=1}^n B^t(i, k)A^t(k, j) = (B^t A^t)(i, j). \end{aligned}$$

(d) Assume $A \in M_n(\mathbb{F})$.

To show: $(A^t)^t = A$.

To show: If $i, j \in \{1, \dots, n\}$ then $((A^t)^t)(i, j) = A(i, j)$.

Assume $i, j \in \{1, \dots, n\}$.

To show: $((A^t)^t)(i, j) = A(i, j)$.

$$((A^t)^t)(i, j) = (A^t)(j, i) = A(i, j)$$

□