

1 Definitions and results

1.1 Matrices and operations: Definitions and results

1.1.1 Matrices

Let $\mathbb{F}$ be a field. Let $m, n \in \mathbb{Z}_{>0}$.

- An $m \times n$ matrix with entries in $\mathbb{F}$ is a table of elements of $\mathbb{F}$ with m rows and n columns. More precisely, an $m \times n$ matrix with entries in $\mathbb{F}$ is a function

$$A: \{1, \dots, m\} \times \{1, \dots, n\} \longrightarrow \mathbb{F}.$$

- A *column vector of length n* is an $n \times 1$ matrix.
- A *row vector of length n* is an $1 \times n$ matrix.
- The (i, j) *entry of a matrix A* is the element $A(i, j)$ in row i and column j of A .

$$A = \begin{pmatrix} A(1, 1) & A(1, 2) & \cdots & A(1, m) \\ A(2, 1) & A(2, 2) & \cdots & A(2, m) \\ \vdots & & & \vdots \\ A(n, 1) & A(n, 2) & \cdots & A(n, m) \end{pmatrix}$$

Let

$M_{m \times n}(\mathbb{F})$ denote the set of $m \times n$ matrices with entries in $\mathbb{F}$.

1.1.2 Addition and scalar multiplication

Let $m, n \in \mathbb{Z}_{>0}$.

- The *sum* of $m \times n$ matrices A and B is the $m \times n$ matrix $A + B$ given by

$$(A + B)(i, j) = A(i, j) + B(i, j), \quad \text{for } i \in \{1, \dots, m\} \text{ and } j \in \{1, \dots, n\}.$$

- The *scalar multiplication* of an element $c \in \mathbb{F}$ with an $m \times n$ matrix A is the $m \times n$ matrix $c \cdot A$ given by

$$(c \cdot A)(i, j) = c \cdot A(i, j), \quad \text{for } i \in \{1, \dots, m\} \text{ and } j \in \{1, \dots, n\}.$$

The *zero matrix* is the $m \times n$ matrix $0 \in M_{m \times n}(\mathbb{F})$ given by

$$0(i, j) = 0, \quad \text{for } i \in \{1, \dots, m\} \text{ and } j \in \{1, \dots, n\}.$$

The *negative* of a matrix $A \in M_{m \times n}(\mathbb{F})$ is the matrix $-A \in M_{m \times n}(\mathbb{F})$ given by

$$(-A)(i, j) = -A(i, j), \quad \text{for } i \in \{1, \dots, m\} \text{ and } j \in \{1, \dots, n\}.$$

Proposition 1.1. Let $m, n \in \mathbb{Z}_{>0}$ and let $M_{m \times n}(\mathbb{F})$ be the set of $m \times n$ matrices with entries in $\mathbb{F}$.

- If $A, B, C \in M_{m \times n}(\mathbb{F})$ then $A + (B + C) = (A + B) + C$.
- If $A, B \in M_{m \times n}(\mathbb{F})$ then $A + B = B + A$.
- If $A \in M_{m \times n}(\mathbb{F})$ then $0 + A = A$ and $A + 0 = A$.
- If $A \in M_{m \times n}(\mathbb{F})$ then $(-A) + A = 0$ and $A + (-A) = 0$.
- If $A \in M_{m \times n}(\mathbb{F})$ and $c_1, c_2 \in \mathbb{F}$ then $c_1 \cdot (c_2 \cdot A) = (c_1 c_2) \cdot A$.
- If $A \in M_{m \times n}(\mathbb{F})$ and $1 \in \mathbb{F}$ is the identity in $\mathbb{F}$ then $1 \cdot A = A$.

1.1.3 Multiplication

- The *product* of an $m \times n$ matrix A and an $n \times p$ matrix B is the $m \times p$ matrix AB given by

$$\begin{aligned}(AB)(i, k) &= \sum_{j=1}^n A(i, j)B(j, k) \\ &= A(i, 1)B(1, k) + A(i, 2)B(2, k) + \cdots + A(i, n)B(n, k),\end{aligned}$$

for $i \in \{1, \dots, m\}$ and $k \in \{1, \dots, p\}$.

The *Kronecker delta* is given by

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

The *identity matrix* is the $n \times n$ matrix $1 \in M_{n \times n}(\mathbb{F})$ given by

$$1(i, j) = \delta_{ij}, \quad \text{for } i \in \{1, \dots, m\} \text{ and } j \in \{1, \dots, n\}.$$

Proposition 1.2. Let $n \in \mathbb{Z}_{>0}$ and let $M_n(\mathbb{F})$ be the set of $n \times n$ matrices in $\mathbb{F}$.

- (a) If $A, B, C \in M_n(\mathbb{F})$ then $A + (B + C) = (A + B) + C$.
- (b) If $A, B \in M_n(\mathbb{F})$ then $A + B = B + A$.
- (c) If $A \in M_n(\mathbb{F})$ then $0 + A = A$ and $A + 0 = A$.
- (d) If $A \in M_n(\mathbb{F})$ then $(-A) + A = 0$ and $A + (-A) = 0$.
- (e) If $A, B, C \in M_n(\mathbb{F})$ then $A(BC) = (AB)C$.
- (f) If $A, B, C \in M_n(\mathbb{F})$ then $(A + B)C = AC + BC$ and $C(A + B) = CA + CB$.
- (g) If $A \in M_n(\mathbb{F})$ then $1A = A$ and $A1 = A$.

1.1.4 Singlun matrices E_{ij}

For $k \in \{1, \dots, m\}$ and $\ell \in \{1, \dots, n\}$ let $E_{k\ell} \in M_{m \times n}(\mathbb{F})$ be the matrix given by

$$E_{k\ell}(i, j) = \begin{cases} 1, & \text{if } i = k \text{ and } j = \ell, \\ 0, & \text{otherwise,} \end{cases}$$

so that $E_{k\ell}$ has a 1 in the (k, ℓ) entry and all other entries 0.

Proposition 1.3. Let $m, n \in \mathbb{Z}_{>0}$, let $M_{m \times n}(\mathbb{F})$ be the set of $m \times n$ matrices with entries in $\mathbb{F}$. Then

- (a) (span) $M_{m \times n}(\mathbb{F}) = \left\{ \sum_{i=1}^m \sum_{j=1}^n c_{ij} E_{ij} \mid c_{ij} \in \mathbb{F} \right\}$.
- (b) (linear independence) If $c_{11}, \dots, c_{mn} \in \mathbb{F}$ and

$$\sum_{i=1}^m \sum_{j=1}^n c_{ij} E_{ij} = 0 \quad \text{then} \quad \text{if } k \in \{1, \dots, m\} \text{ and } \ell \in \{1, \dots, n\} \text{ then } c_{k\ell} = 0.$$

1.1.5 Matrix transpose

The *transpose* of an $m \times n$ matrix A is the $n \times m$ matrix A^t given by

$$A^t(i, j) = A(j, i), \quad \text{for } i \in \{1, \dots, n\} \text{ and } j \in \{1, \dots, m\}.$$

Proposition 1.4. *Let $m, n \in \mathbb{Z}_{>0}$, let $M_{m \times n}(\mathbb{F})$ be the set of $m \times n$ matrices with entries in $\mathbb{F}$, and let $M_n(\mathbb{F})$ be the set of $n \times n$ matrices in $\mathbb{F}$.*

- (a) *If $A, B \in M_{m \times n}(\mathbb{F})$ then $(A + B)^t = A^t + B^t$,*
- (b) *If $A \in M_{m \times n}(\mathbb{F})$ and $c \in \mathbb{F}$ then $(c \cdot A)^t = c \cdot A^t$,*
- (c) *If $A, B \in M_n(\mathbb{F})$ then $(AB)^t = B^t A^t$.*
- (d) *If $A \in M_n(\mathbb{F})$ then $(A^t)^t = A$.*