Matrices

A *matrix* is a table of numbers

$$A = \begin{pmatrix} 78 & 62 & 91 & 85 \\ 32 & 41 & 24 & 39 \\ 6 & 99 & 29 & 81 \end{pmatrix}$$

Addition

$$\underset{A}{\begin{pmatrix} 78 & 62 & 91 & 85 \\ 32 & 41 & 24 & 39 \\ 6 & 99 & 29 & 81 \end{pmatrix}} + \underset{B}{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ -1 & -2 & -3 & -4 \end{pmatrix}} = \underset{A+B}{\begin{pmatrix} 79 & 64 & 94 & 89 \\ 37 & 47 & 31 & 47 \\ 5 & 97 & 26 & 77 \end{pmatrix}}$$

the (i,j) entry of $A+B$ is
(the (i,j) entry of A) + (the (i,j) entry of B).

Scalar multiplication

$$\underset{\frac{1}{3}\ \cdot\ A}{\frac{1}{3}\begin{pmatrix} 78 & 62 & 91 & 85 \\ 32 & 41 & 24 & 39 \\ 6 & 99 & 29 & 81 \end{pmatrix}} = \underset{\frac{1}{3}A}{\begin{pmatrix} 26 & \frac{62}{3} & 30\frac{1}{3} & \frac{85}{3} \\ 10\frac{2}{3} & \frac{41}{3} & 8 & 13 \\ 2 & 33 & 7\frac{2}{3} & 27 \end{pmatrix}}$$

the (i,j) entry of cA is
$c\cdot$ (the (i,j) entry of A.

Let $t, s \in \mathbb{Z}_{>0}$.

$$M_{t\times s}(\mathbb{Q}) = \left\{ \begin{array}{c} t\times s \text{ matrices with} \\ \text{entries in } \mathbb{Q} \end{array} \right\}$$

$$M_{2\times 3}(\mathbb{Q}) = \left\{ \begin{array}{c} 2\times 3 \text{ matrices with} \\ \text{entries in } \mathbb{Q} \end{array} \right\}$$

Matrix units

Let E_{ij} be with 1 in the (i,j) entry and 0 elsewhere.

$$E_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$E_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad E_{22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad E_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 78 & 62 & 91 \\ 32 & 41 & 24 \end{pmatrix} = 78E_{11} + 62E_{12} + 91E_{13} + 32E_{21} + 41E_{22} + 24E_{23}$$

Theorem Every matrix is a (unique) linear combination of the E_{ij}.

$\dim(M_{2\times 3}(\mathbb{Q})) = 6$,

because there are 6 E_{ij}s in this case

$\{E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23}\}$.

Multiplication

Example $(2\ 5\ 11\ 13)\begin{pmatrix}4\\0\\3\\-2\end{pmatrix} = 2\cdot 4+5\cdot 0+11\cdot 3+13\cdot(-2)$
$$= 8+0+33-26 = 15.$$

Example
$$\underset{A}{\begin{pmatrix}2 & 5 & 11 & 13\\ 1 & 2 & -1 & 0\end{pmatrix}}\underset{B}{\begin{pmatrix}4 & -1\\ 0 & 6\\ 3 & 0\\ 2 & 1\end{pmatrix}} = \begin{pmatrix}15 & -2+30+13\\ 4-3 & -1+12\end{pmatrix}$$
$$= \underset{AB}{\begin{pmatrix}15 & 41\\ 1 & 11\end{pmatrix}}$$

The (i,j) entry of AB is
(the i^{th} row of A) times (the j^{th} column of B)

Example Let $A=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ and $B=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$.

$$AB = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix} = \begin{pmatrix}1+1 & 0+1\\ 0+1 & 0+1\end{pmatrix} = \begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}$$

$$BA = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1+0 & 1+0\\ 1+0 & 1+1\end{pmatrix} = \begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}$$

So $AB \neq BA$ in this example.

Identity matrix

For $n=2$: $1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ in $M_{2\times 2}(\mathbb{Q})$

For $n=3$: $1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ in $M_{3\times 3}(\mathbb{Q})$

For $n=4$: $1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ in $M_{4\times 4}(\mathbb{Q})$

In general,

$$1 = E_{11} + E_{22} + \cdots + E_{nn} \text{ in } M_{n\times n}(\mathbb{Q})$$

The set of invertible $n\times n$ matrices is

$$GL_n(\mathbb{Q}) = \left\{ A \in M_{n\times n}(\mathbb{Q}) \;\middle|\; \begin{array}{l} \text{there exists } A^{-1} \in M_{n\times n}(\mathbb{Q}) \\ \text{such that} \\ AA^{-1} = 1 \text{ and } A^{-1}A = 1 \end{array} \right\}$$

Some easy inverses

$x_{12}(c) = \begin{pmatrix} 1 & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ has $x_{12}(c)^{-1} = \begin{pmatrix} 1 & -c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

since

$$\begin{pmatrix} 1 & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1$$

$$\begin{pmatrix} 1 & -c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1$$

06.01.2024 (5)

Lin Alg Lect. 10 A. Ram

$$x_{13}(c) = \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ has } x_{13}(c)^{-1} = \begin{pmatrix} 1 & 0 & -c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$x_{23}(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \text{ has } x_{23}(c)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix}$$

<u>Diagonal generators</u>

$$h_1\left(\tfrac{a}{b}\right) = \begin{pmatrix} \frac{a}{b} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ has } h_1\left(\tfrac{a}{b}\right)^{-1} = \begin{pmatrix} \frac{b}{a} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$h_2\left(\tfrac{a}{b}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{a}{b} & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ has } h_2\left(\tfrac{a}{b}\right)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & b/a & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$h_3\left(\tfrac{a}{b}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a/b \end{pmatrix} \text{ has } h_3\left(\tfrac{a}{b}\right)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & b/a \end{pmatrix}$$

<u>Row reducers</u>

$$s_1(c) = \begin{pmatrix} c & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ has } s_1(c)^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -c & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$s_2(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & 1 \\ 0 & 1 & 0 \end{pmatrix} \text{ has } s_2(c)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -c \end{pmatrix}$$