MAST10007 Linear Algebra

THE UNIVERSITY OF MELBOURNE
SCHOOL OF MATHEMATICS AND STATISTICS

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Arun Ram: Additional Slides

These slides have been made by Arun Ram, in preparation for teaching of the summer session of MAST10007 Linear Algebra at University of Melbourne in 2026. The template is from the University of Melbourne School of Mathematics and Statistics slide deck which was produced by members of the School including, in particular, huge developments by Craig Hodgson and Christine Mangelsdorf.

Lecture 1: Matrices

A matrix is a table of numbers.

$$A = \begin{pmatrix} 78 & 62 & 91 & 85 \\ 32 & 41 & 24 & 39 \\ 6 & 99 & 29 & 81 \end{pmatrix}$$

Some applications of matrices are

1. Solving systems of linear equations
2. lengths, distances, angles, projections
3. Equations of lines and planes, volumes of parallelipipeds
4. graphs and networks
5. Data processing and analysis of data
6. Dynamics
7. Symmetry
8. Quantum mechanics
9. ... and many many more ...

Addition

$$\begin{pmatrix} 78 & 62 & 91 & 85 \\ 32 & 41 & 24 & 39 \\ 6 & 99 & 29 & 81 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ -1 & -2 & -3 & -4 \end{pmatrix} = \begin{pmatrix} 79 & 64 & 94 & 89 \\ 37 & 48 & 31 & 47 \\ 5 & 97 & 26 & 77 \end{pmatrix}$$

Scalar multiplication

$$\frac{1}{3}\begin{pmatrix} 78 & 62 & 91 & 85 \\ 32 & 41 & 24 & 39 \\ 6 & 99 & 29 & 81 \end{pmatrix} = \begin{pmatrix} 26 & \frac{62}{3} & 27 & \frac{85}{3} \\ 10\frac{2}{3} & \frac{41}{3} & 8 & 13 \\ 2 & 33 & \frac{29}{3} & 27 \end{pmatrix}$$

Definition (Matrix units)

Let $t, s \in \mathbb{Z}_{>0}$ and let $i \in \{1, \ldots, t\}$ and $j \in \{1, \ldots, s\}$. The *matrix unit E_{ij}* is the matrix

$$E_{ij} \in M_{t \times s}(\mathbb{Q}) \quad \text{which has} \quad \begin{array}{l} 1 \text{ in the } (i,j)\text{-entry} \\ \text{and } 0 \text{ elsewhere,} \end{array}$$

The favourite basis of $M_{t\times s}(\mathbb{Q})$

If $t = 2$ and $s = 3$ then

$$E_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$E_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad E_{22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad E_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Every matrix is a (unique) linear combination of E_{ij}

('linear' means using scalar multiplication and addition).

$$\begin{pmatrix} 78 & 62 & 91 \\ 32 & 41 & 24 \end{pmatrix} = \begin{array}{r} 78E_{11} + 62E_{12} + 91E_{13} \\ +32E_{21} + 41E_{22} + 24E_{23} \end{array}$$

Multiplication

The (i,j) entry of AB is the ith row of A times the jth column of B.

$$\begin{pmatrix} 2 & 5 & 11 & 13 \end{pmatrix} \begin{pmatrix} 4 \\ 0 \\ 3 \\ -2 \end{pmatrix} = 2 \cdot 4 + 5 \cdot 0 + 11 \cdot 3 + 13 \cdot (-2)$$

$$= 8 + 33 - 26 = 15.$$

$$\begin{pmatrix} 78 & 62 & 91 & 85 \\ 32 & 41 & 24 & 39 \\ 6 & 99 & 29 & 81 \end{pmatrix} \begin{pmatrix} \frac{2}{100} \\ \frac{85}{100} \\ \frac{1}{100} \\ \frac{12}{100} \end{pmatrix} = \begin{pmatrix} 65.37 \\ 40.41 \\ 94.28 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 0 & 43 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -43 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Theorem (Properties of matrix operations)

1. *If* $A, B \in M_{t\times s}(\mathbb{Q})$ *then* $A + B = B + A$*.*
2. *If* $A, B, C \in M_{t\times s}(\mathbb{Q})$ *then* $A + (B + C) = (A + B) + C$*.*
3. *If* $A \in M_{t\times s}(\mathbb{Q})$*,* $B \in M_{s\times r}(\mathbb{Q})$ *and* $C \in M_{r\times q}(\mathbb{Q})$ *then*
$$A(BC) = (AB)C.$$
4. *If* $A, B \in M_{t\times s}(\mathbb{Q})$ *and* $C, D \in M_{s\times r}(\mathbb{Q})$ *then*
$$A(C + D) = AC + AD \quad \text{and} \quad (A + B)C = AC + BC.$$
5. *If* $A \in M_{t\times s}(\mathbb{Q})$*,* $B \in M_{s\times r}(\mathbb{Q})$ *and* $c \in \mathbb{Q}$ *then* $A(cB) = c(AB)$*.*
6. *If* $A \in M_{t\times s}(\mathbb{Q})$ *and* 1 *is the identity in* $M_{s\times s}(\mathbb{Q})$ *then* $A \cdot 1 = A$*.*
7. *If* $A \in M_{t\times s}(\mathbb{Q})$ *and* 1 *is the identity in* $M_{t\times t}(\mathbb{Q})$ *then* $1 \cdot A = A$*.*
8. *If* $A \in M_{t\times s}(\mathbb{Q})$ *then* $A + 0 = A$ *and* $0 + A = A$*.*

Warning. The list of properties of matrix operations says that for the most part the matrix number system works much like the ordinary integer number system. But be careful.

$$\text{If} \quad A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

then

$$AB = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

and

$$BA = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

and so AB is *not the same* as BA. For most matrices A and B, the product AB is not the same as BA. When it does happen, that should be viewed as very special and very lucky. Don't push your luck.

Favourite square matrices

Definition (Invertible matrices)

Let $n \in \mathbb{Z}_{>0}$. Let E_{ij} be the $n \times n$ matrix with 1 in the (i,j) entry and 0 elswhere. The *identity matrix* is

$$1 = E_{11} + \cdots + E_{nn} \quad \text{in } M_{n\times n}(\mathbb{Q}).$$

The set of invertible $n \times n$ matrices is

$$GL_n(\mathbb{Q}) = \left\{ A \in M_{n\times n}(\mathbb{Q}) \;\middle|\; \begin{array}{l} \text{there exists } A^{-1} \in M_{n\times n}(\mathbb{Q}) \\ \text{such that } AA^{-1} = 1 \text{ and } A^{-1}A = 1. \end{array} \right\}$$

If $n = 2$ then

$$\begin{pmatrix} c & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -c \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$$

and

$$\begin{pmatrix} 0 & 1 \\ 1 & -c \end{pmatrix} \begin{pmatrix} c & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1.$$

Example A1. (Root matrices and their inverses) If $c \in \mathbb{Q}$ and

$$x_{12}(c) = \begin{pmatrix} 1 & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x_{13}(c) = \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x_{23}(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

then

$$x_{12}(c)^{-1} = \begin{pmatrix} 1 & -c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x_{13}(c)^{-1} = \begin{pmatrix} 1 & 0 & -c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$x_{23}(c)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix}.$$

Example A2. (Diagonal generators and their inverses) If $d \in \mathbb{Q}$ and $d \neq 0$ and

$$h_1(d) = \begin{pmatrix} d & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; h_2(d) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; h_3(d) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & d \end{pmatrix}$$

then

$$h_1(d)^{-1} = \begin{pmatrix} \frac{1}{d} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad h_2(d)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{d} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$h_3(d)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{d} \end{pmatrix}.$$

Example A3. (Row reducers and their inverses.) If $c \in \mathbb{Q}$ then

$$s_1(c) = \begin{pmatrix} c & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad s_2(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

then

$$s_1(c)^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -c & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad s_2(c)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -c \end{pmatrix}$$

Let $n \in \mathbb{Z}_{>0}$. Let E_{ij} be the $n \times n$ matrix with 1 in the (i,j) entry and 0 elswhere.

Definition (root matrices, diagonal generators and row reducers)

Let $i, j \in \{1, \ldots, n\}$ with $i \neq j$. Let $c \in \mathbb{Q}$. The *root matrix $x_{ij}(c)$* is

$$x_{ij}(c) \in M_{n\times n}(\mathbb{Q}) \quad \text{given by} \quad x_{ij}(c) = 1 + cE_{ij}.$$

Let $i \in \{1, \ldots, n\}$. Let $d \in \mathbb{Q}$ with $d \neq 0$. The *diagonal generator $h_i(d)$* is

$$h_i(d) = 1 + (d-1)E_{ii}.$$

Let $i \in \{1, \ldots, n-1\}$ and let $c \in \mathbb{Q}$. The *row reducer $s_i(c)$* is

$$s_i(c) = 1 - E_{ii} - E_{i+1,i+1} + E_{i,i+1} + E_{i+1,i} + cE_{ii}.$$

Theorem (Generators for GL_n)

Let $A \in GL_n(\mathbb{Q})$. Then A can be written as a product of row reducers, diagonal generators and root matrices.