Lecture 19: Eigenvalues and eigenvectors

Definition (Eigenvectors and eigenvalues.)

Let $A \in M_n(\mathbb{Q})$.

- An *eigenvalue of A* is an element $\lambda \in \mathbb{Q}$ such that $\ker(A - \lambda) \neq 0$.

Let $A \in M_n(\mathbb{Q})$ and $\lambda \in \mathbb{Q}$.

- An *eigenvector of A of eigenvalue λ* is a nonzero element of $\ker(A - \lambda)$.

Definition (Eigenvectors, eigenvalues and diagonalization.)

Let $f : V \to V$ be an $\mathbb{F}$-linear transformation.

- An *eigenvalue of f* is an element $\lambda \in \mathbb{F}$ such that $\ker(A - \lambda) \neq 0$.

Let $f : V \to V$ be an $\mathbb{F}$-linear transformation and let $\lambda \in \mathbb{F}$.

- An *eigenvector of f of eigenvalue λ* is a nonzero element of $\ker(f - \lambda)$.

Definition (Eigenvectors and eigenvectors of a matrix.)

Let $A \in M_n(\mathbb{Q})$.

- An *eigenvalue of A* is an element $\lambda \in \mathbb{Q}$ such that $\ker(A - \lambda) \neq 0$.

Let $A \in M_n(\mathbb{Q})$ and $\lambda \in \mathbb{Q}$.

- An *eigenvector of A of eigenvalue λ* is a nonzero element of $\ker(A - \lambda)$.

Theorem (Diagonalization.)

Let $A \in M_{n\times n}(\mathbb{F})$. The matrix A has n linearly independent eigenvectors $\mathbf{p}_1, \dots, \mathbf{p}_n \in \mathbb{F}^n$ with eigenvalues $\lambda_1, \dots, \lambda_n$ if and only if $A = PDP^{-1}$ where,

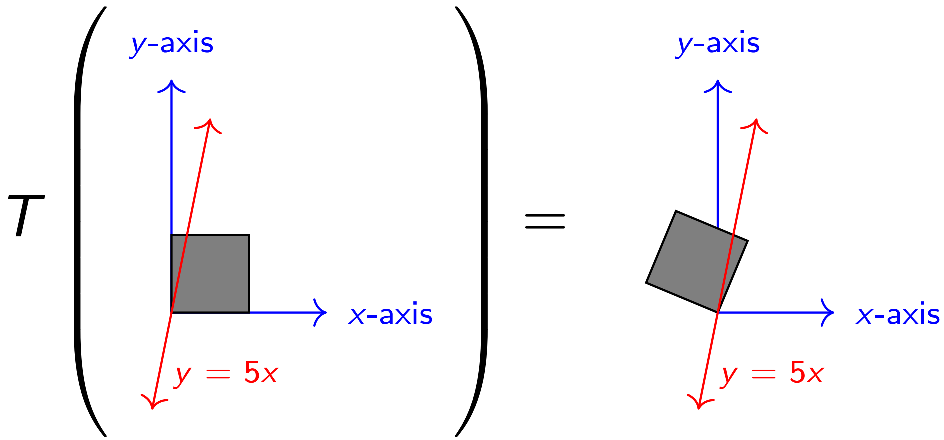
$$P = \begin{pmatrix} | & & | \\ \mathbf{p}_1 & \cdots & \mathbf{p}_n \\ | & & | \end{pmatrix} \quad \text{and} \quad D = \operatorname{diag}(\lambda_1, \dots, \lambda_n) = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

so that $\mathbf{p}_1, \dots, \mathbf{p}_n$ are the columns of P and D is the diagonal matrix with diagonal entries $\lambda_1, \dots, \lambda_n$.

If **v** is an eigenvector of A of eigenvalue λ then $(A - \lambda)\mathbf{v} = 0$ and

$$A\mathbf{v} = \lambda\mathbf{v}.$$

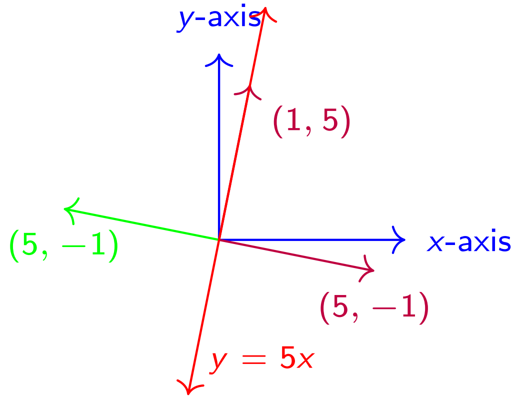
Example EV1. Let $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection in the line $y = 5x$.



Identify two lines through the origin that are invariant under T and find the image of the direction vectors for each of these lines.

Let

$$B = \{(1,5),(5,-1)\},$$



One line is the line $y = 5x$ and the other line is the line orthogonal to $y = 5x$. The line $y = 5x$ has slope 5 and the line orthogonal to $y = 5x$ has slope $-\frac{1}{5}$ and equation $y = -\frac{1}{5}x$. The corresponding direction vectors of these lines are $(1,5)$ and $(1,-\frac{1}{5})$ and

$$T(1,5) = (1,5) \qquad \text{and} \qquad T(1,-\tfrac{1}{5}) = -(1,-\tfrac{1}{5}) = (-1,\tfrac{1}{5}).$$

If $\mathbf{v}_1 = (1,5)$ and $v_2 = (1,-\frac{1}{5})$ then

$$Tv_1 = 1 \cdot v_1 \quad \text{and} \quad Tv_2 = (-1) \cdot v_2.$$

Example EV2,6&9. Find the eigenvalues of $A = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$.

First,

$$\begin{aligned} A - t &= \begin{pmatrix} 1-t & 4 \\ 1 & 1-t \end{pmatrix} = \begin{pmatrix} 1-t & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1-t \\ 0 & 4-(1-t)^2 \end{pmatrix} \\ &= \begin{pmatrix} 1-t & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1-t \\ 0 & -(t+1)(t-3) \end{pmatrix} \end{aligned}$$

Case 1: $t+1=0$. Then

$$A+1 = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \ker(A+1) = \text{span}\left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\}$$

Case 1: $t-3=0$. Then

$$A-3 = \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \ker(A-3) = \text{span}\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}.$$

If $\quad P = \begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$ then $P^{-1} = \dfrac{1}{4} \begin{pmatrix} -1 & 2 \\ 1 & 2 \end{pmatrix}$

$$\begin{aligned} \text{and} \quad PDP^{-1} &= \begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} \cdot \tfrac{1}{4} \begin{pmatrix} -1 & 2 \\ 1 & 2 \end{pmatrix} \\ &= \tfrac{1}{4} \begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 3 & 6 \end{pmatrix} \\ &= \tfrac{1}{4} \begin{pmatrix} 4 & 16 \\ 4 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} = A. \end{aligned}$$

The characteristic polynomial of A is

$$\det(A - t) = \det(D - t) = (-1 - t)(3 - t).$$

Example EV3,4&10. Find the eigenvalues of $A = \begin{pmatrix} 2 & -3 & 6 \\ 0 & 5 & -6 \\ 0 & 1 & 0 \end{pmatrix}$.

Find $\ker(A - t)$ by row reduction:

$$A - t = \begin{pmatrix} 2-t & -3 & 6 \\ 0 & 5-t & -6 \\ 0 & 1 & 0-t \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5-t & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2-t & -3 & 6 \\ 0 & 1 & -t \\ 0 & 0 & -6-(5-t)(-t) \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5-t & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2-t & 0 & 3(2-t) \\ 0 & 1 & -t \\ 0 & 0 & -(t^2-5t+6) \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 3 & 0 \\ 0 & 5-t & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2-t & 0 & 3(2-t) \\ 0 & 1 & -t \\ 0 & 0 & -(t-2)(t-3) \end{pmatrix}.$$

Case 1: $t-2=0$. Then

$$A-2=\begin{pmatrix}1 & 3 & 0\\ 0 & 5-2 & 1\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 1 & -2\\ 0 & 0 & 0\end{pmatrix}$$

and

$$\ker(A-2)=\operatorname{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\2\\1\end{pmatrix}\right\}.$$

Case 2: $t-3=0$. Then

$$\begin{aligned}A-3&=\begin{pmatrix}1 & 3 & 0\\ 0 & 5-3 & 1\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}2-3 & 0 & 3(2-3)\\ 0 & 1 & -3\\ 0 & 0 & 0\end{pmatrix}\\ &=\begin{pmatrix}1 & 3 & 0\\ 0 & 2 & 1\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}-1 & 0 & -3\\ 0 & 1 & -3\\ 0 & 0 & 0\end{pmatrix}\end{aligned}$$

and

$$\ker(A-3) = \operatorname{span}\left\{ \begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix} \right\}.$$

Then $A = PDP^{-1}$ where

$$P = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 2 & 3 \\ 0 & 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix},$$

$$\text{and} \quad P^{-1} = - \begin{pmatrix} -1 & 0 & 0 \\ -3 & 1 & -1 \\ 6 & -3 & 2 \end{pmatrix}^t = \begin{pmatrix} 1 & 3 & -6 \\ 0 & -1 & 3 \\ 0 & 1 & -2 \end{pmatrix}.$$

The characteristic polynomial of A is

$$\det(A-t) = \det(D-t) = (2-t)^2(3-t).$$

Example 5,8&12. As an element of $M_{2\times 2}(\mathbb{R})$, the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{has no eigenvalues and no eigenvectors.}$$

The linear transformation

$$\begin{array}{rccc} T: & \mathbb{R}^2 & \to & \mathbb{R}^2 \\ & v & \mapsto & Av \end{array} \quad \text{is a rotation of } \tfrac{3\pi}{2} \text{ about } (0,0).$$

As an element of $M_{2\times 2}(\mathbb{C})$, the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{has two eigenvalues, } i \text{ and } -i.$$

$$\ker(A - i) = \text{span}\left\{\begin{pmatrix} i \\ 1 \end{pmatrix}\right\} \qquad \text{and} \qquad \ker(A + i) = \text{span}\left\{\begin{pmatrix} -i \\ 1 \end{pmatrix}\right\}.$$

$\begin{pmatrix} i \\ 1 \end{pmatrix}$ is an eigenvector of eigenvalue i and
$\begin{pmatrix} -i \\ 1 \end{pmatrix}$ is an eigenvector of eigenvalue $-i$.

If

$$P = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} \text{ and } D = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \text{ then } P^{-1} = \frac{1}{-2i} \begin{pmatrix} 1 & -i \\ -1 & -i \end{pmatrix}$$

and

$$\begin{aligned} PDP^{-1} &= \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \cdot \frac{1}{-2i} \begin{pmatrix} 1 & -i \\ -1 & -i \end{pmatrix} \\ &= \tfrac{1}{2} i \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} i & 1 \\ i & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = A. \end{aligned}$$

The characteristic polynomial of A is

$$\det(A - t) = \det(D - t) = (i - t)(-i - t) = t^2 + 1.$$

Example EV11. If $PDP^{-1} = A$ then the columns of P are linearly independent eigenvectors of A. Here is an example where A does not have n linearly independent eigenvectors.

$$\text{If} \qquad A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \qquad \text{then} \qquad A - t = \begin{pmatrix} 1-t & 2 \\ 0 & 1-t \end{pmatrix}$$

which has a single row of 0s when $t = 1$.
(The characteristic polynomial of A is $\det(A - t) = (1-t)^2$.)

$$\ker(A - 1) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}.$$

Since A does not have two linearly independent eigenvectors then

A is not diagonalizable.