1.18 Pop quiz on Lecture 18 material

1. Let $t, s \in \mathbb{Z}_{>0}$ and let $A \in M_{t\times s}(\mathbb{Q})$.
   (a) Carefully define $\ker(A)$.
   (b) Carefully define $\mathrm{im}(A)$.
   (c) Carefully define the column space of A.
   (d) Explain (with proof) the relationship between $\mathrm{im}(A)$ and the column space of A.

2. Let $t, s \in \mathbb{Z}_{>0}$ and let $A \in M_{t\times s}(\mathbb{Q})$. Let $P \in M_{t\times t}(\mathbb{Q})$ be an invertible $t \times t$ matrix and let $Q \in M_{s\times s}(\mathbb{Q})$ be an invertible $s \times s$ matrix.
   (a) Prove carefully that $\ker(PA) = \ker(A)$.
   (b) Prove carefully that $\mathrm{im}(AQ) = \mathrm{im}(A)$.

3. Let $t, s \in \mathbb{Z}_{>0}$ and let $A \in M_{t\times s}(\mathbb{Q})$. Let $P \in M_{t\times t}(\mathbb{Q})$ be an invertible $t \times t$ matrix and let $Q \in M_{s\times s}(\mathbb{Q})$ be an invertible $s \times s$ matrix.
   (a) Prove carefully that $\ker(QA) = Q^{-1}\ker(A)$.
   (b) Prove carefully that $\mathrm{im}(PA) = P\mathrm{im}(A)$.

4. Let $t, s \in \mathbb{Z}_{>0}$ and let $A \in M_{t\times s}(\mathbb{Q})$. Let $r \in \{1, \ldots, \min(s,t)\}$ and let

   $$1_r = E_{11} + \cdots + E_{rr} \quad \text{in } M_{t\times s}(\mathbb{Q}),$$

   where E_{ij} has 1 in the (i,j) entry and 0 elsewhere. Let $e_i \in \mathbb{Q}^s$ be the vector with 1 in the ith entry and 0 elsewhere so that

   $$S = \{e_1, \ldots, e_s\} \quad \text{is the standard basis of } \mathbb{Q}^s.$$

   (a) Prove carefully that $\ker(1_r) = \mathbb{Q}\text{-span}\{e_{r+1}, \ldots, e_s\}$.
   (b) Let $Q \in GL_s(\mathbb{Q})$ so that Q is an invertible $s \times s$ matrix. Prove carefully that

   $$\ker(1_r Q) = \mathbb{Q}\text{-span}\{\text{last } s-r \text{ columns of } Q^{-1}\}.$$

   (c) Prove carefully that $\mathbb{Q}$-span{last $s-r$ columns of Q^{-1}} is a basis of $\ker(1_r Q)$.

5. Let $t, s \in \mathbb{Z}_{>0}$ and let $A \in M_{t\times s}(\mathbb{Q})$. Let $r \in \{1, \ldots, \min(s,t)\}$ and let

   $$1_r = E_{11} + \cdots + E_{rr} \quad \text{in } M_{t\times s}(\mathbb{Q}),$$

   where E_{ij} has 1 in the (i,j) entry and 0 elsewhere. Let $e_i \in \mathbb{Q}^t$ be the vector with 1 in the ith entry and 0 elsewhere so that

   $$B = \{e_1, \ldots, e_t\} \quad \text{is the standard basis of } \mathbb{Q}^t.$$

   (a) Prove carefully that $\mathrm{im}(1_r) = \mathbb{Q}\text{-span}\{e_1, \ldots, e_r\}$.
   (b) Let $P \in GL_t(\mathbb{Q})$ so that P is an invertible $t \times t$ matrix. Prove carefully that

   $$\mathrm{im}(P1_r) = \mathbb{Q}\text{-span}\{\text{first } r \text{ columns of } P\}.$$

   (c) Prove carefully that $\mathbb{Q}$-span{first r columns of P} is a basis of $\mathrm{im}(P1_r)$.

6. Let $S = \{|1,3,-1,1\rangle, |2,6,0,4\rangle, |3,9,-2,4\rangle\}$ in $\mathbb{R}^4$. Let

$$P = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 3 & 6 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}, \quad 1_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(a) Compute $A = P1_2Q$.

(b) Show that $\mathbb{Q}$-span$(S) = \mathrm{im}(A)$.

(c) Find a basis of $\mathrm{im}(A)$.

(d) Find a basis of $\ker(A)$.