Lect 18 A. Ram
Linear algebra.

Let A be a 4×5 matrix, $A\in M_{4\times 5}(\mathbb{Q})$

$$\ker(A)=\{\vec{x}\in\mathbb{Q}^5 \mid A\vec{x}=0\}$$

$$\operatorname{im}(A)=\{A\vec{x}\mid \vec{x}\in\mathbb{Q}^5\}.$$

Let $T: V\to W$ be a linear transformation

$$\ker(T)=\{v\in V\mid T(v)=0\}.$$

$$\operatorname{im}(T)=\{T(v)\mid v\in V\}.$$

Let $A\in M_{4\times 5}(\mathbb{Q})$ and $P\in GL_4(\mathbb{Q})$.
Then

$$\ker(PA)=\{\vec{x}\in\mathbb{Q}^5\mid PA\vec{x}=0\}$$

$$=\{x\in\mathbb{Q}^5\mid P^{-1}PA\vec{x}=P^{-1}0\}.$$

$$=\{x\in\mathbb{Q}^5\mid A\vec{x}=0\}$$

$$=\ker(A)$$

Let $Q\in GL_5(\mathbb{Q})$. Then

$$\operatorname{im}(AQ)=\{AQ\vec{x}\mid \vec{x}\in\mathbb{Q}^5\}=\{AQ\vec{x}\mid Q\vec{x}\in\mathbb{Q}^5\}$$

$$=\{A\vec{y}\mid \vec{y}\in\mathbb{Q}^5\}=\operatorname{im}(A)$$

Example

Show that $\ker(A)$ is a subspace of $\mathbb{Q}^5$

A subspace of $\mathbb{Q}^5$ is a subset $L \subseteq \mathbb{Q}^5$ such that

(a) $0 \in L$

(b) If $l_1, l_2 \in L$ then $l_1 + l_2 \in L$

(c) If $l \in L$ and $c \in \mathbb{Q}$ then $cl \in L$.

(a) Since $A \cdot 0 = 0$ then $0 \in \ker(A)$.

(b) Assume $\vec{x}, \vec{y} \in \ker(A)$. Then

$$A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y} = \vec{0} + \vec{0} = \vec{0}.$$

So $\vec{x}+\vec{y} \in \ker(A)$.

(c) Assume $\vec{x} \in \ker(A)$ and $c \in \mathbb{Q}$. Then

$$A c\vec{x} = cA\vec{x} = c \cdot \vec{0} = \vec{0}.$$

So $c\vec{x} \in \ker(A)$

So A is a subspace.

Favorite basis of $\mathbb{Q}^4$

$$S = \{e_1, e_2, e_3, e_4\}$$

where e_i has 1 in ith entry and 0 elsewhere.

$$e_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad e_2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad e_3 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad e_4 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

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Let $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in M_{3\times 4}(\mathbb{Q})$.

Then $\ker(A) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \;\middle|\; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

$= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{Q}^4 \;\middle|\; \begin{matrix} x_1 = 0 \\ x_2 = 0 \\ 0 = 0 \\ 0 = 0 \end{matrix} \right\} = \left\{ \begin{pmatrix} 0 \\ 0 \\ x_3 \\ x_4 \end{pmatrix} \;\middle|\; x_3, x_4 \in \mathbb{Q} \right\}$

$= \left\{ x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \;\middle|\; x_3, x_4 \in \mathbb{Q} \right\}$

$= \mathrm{span}\{e_3, e_4\}$.

Let $t, s \in \mathbb{Z}_{>0}$ and $r \in \{1, \ldots, \min(t,s)\}$. Let

$1_r = E_{11} + E_{22} + \cdots + E_{rr}$ in $M_{t\times s}(\mathbb{Q})$

where E_{ij} has 1 in the (i,j) entry and 0 elsewhere.

Then

$\ker(1_r) = \mathrm{span}\{e_{r+1}, \ldots, e_s\}$

where e_j is the $s\times 1$ column vector with 1 in the jth entry and 0 elsewhere.

Let $A \in M_{4\times 5}(\mathbb{Q})$ and let $Q \in GL_5(\mathbb{Q})$ so that Q is an invertible 5×5 matrix.

Then

$$\begin{aligned}
\ker(AQ) &= \{x \in \mathbb{Q}^5 \mid AQ\vec{x} = 0\} \\
&= \{Q^{-1}Q\vec{x} \mid AQ\vec{x} = 0\} \\
&= Q^{-1}\{Q\vec{x} \mid AQ\vec{x} = 0\} \\
&= Q^{-1}\{\vec{y} \mid A\vec{y} = 0\} \\
&= Q^{-1}\ker(A).
\end{aligned}$$

If $A = P 1_r Q$ then

The number r is the <u>rank of A</u>.

$$\begin{aligned}
\ker(A) &= \ker(P 1_r Q) \\
&= \ker(1_r Q) \\
&= Q^{-1}\ker(1_r) \\
&= Q^{-1}\operatorname{span}\{e_{r+1}, \ldots, e_5\} \\
&= \operatorname{span}\{Q^{-1}e_{r+1}, \ldots, Q^{-1}e_5\}
\end{aligned}$$

and $\{Q^{-1}e_{r+1}, \ldots, Q^{-1}e_5\}$ is a basis of $\ker(A)$

So $\dim(\ker(A)) = 5 - r$. (equals # of cols of A − rank(A))

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Let $1_r = E_{11}+E_{22}+\cdots+E_{rr}$ in $M_{4\times 5}(\mathbb{Q})$.

$$\mathrm{im}(1_r) = \{1_r\vec{x} \mid x\in\mathbb{Q}^5\}$$

$$= \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \;\middle|\; x_1,x_2,x_3,x_4,x_5\in\mathbb{Q} \right\}$$

$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ 0 \\ 0 \end{pmatrix} \;\middle|\; x_1,x_2,x_3,x_4,x_5\in\mathbb{Q} \right\}$$

$$= \left\{ x_1\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_2\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \;\middle|\; x_1,x_2\in\mathbb{Q} \right\}$$

$= \mathrm{span}\{e_1,e_2,\ldots,e_r\}$, where

$e_j\in\mathbb{Q}^4$ has 1 in $j^{\underline{th}}$ entry and 0 elsewhere.

Thm: If $A\in M_{4\times5}(\mathbb{Q})$ and $P\in GL_4(\mathbb{Q})$ then

$$\mathrm{im}(PA) = \{PA\vec{x} \mid x\in\mathbb{Q}^5\}$$

$$= P\{A\vec{x} \mid x\in\mathbb{Q}^5\} = P\,\mathrm{im}(A).$$

So, if $A = P1_rQ$ then

$$\mathrm{im}(A) = \mathrm{im}(P1_rQ) = \mathrm{im}(P1_r) = P\,\mathrm{im}(1_r)$$

$= P\,\mathrm{span}\{e_1,\ldots,e_r\} = \mathrm{span}\{Pe_1,\ldots,Pe_r\}$

and $\{Pe_1,\ldots,Pe_r\}$ is a basis of $\mathrm{im}(A)$.