Lecture 18: Row space, column space and solution space

Definition (Row space, column space and solution space)

Let $A \in M_{t \times s}(\mathbb{R})$.

The *column space* of A is the span of the columns of A.

The *row space* of A is the span of the rows of A.

The *kernel of A* is $\ker(A) = \{\mathbf{x} \in \mathbb{R}^s \mid A\mathbf{x} = 0\}$.

The column space of A is the image of A since

$$\mathrm{im}(A) = \{A\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^s\} = \left\{ \begin{pmatrix} | & & | \\ a_1 & \cdots & a_s \\ | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_s \end{pmatrix} \;\middle|\; x_1, \ldots, x_s \in \mathbb{R} \right\}$$

$$= \left\{ x_1 \begin{pmatrix} | \\ a_1 \\ | \end{pmatrix} + \cdots + x_s \begin{pmatrix} | \\ a_s \\ | \end{pmatrix} \;\middle|\; x_1, \ldots, x_s \in \mathbb{R} \right\} = \mathrm{colspace}(A)$$

Let $A \in M_{t\times s}(\mathbb{Q})$ and $P \in GL_t(\mathbb{Q})$ and $Q \in GL_s(\mathbb{Q})$. Then

$$\begin{aligned}
\ker(PA) &= \{x \in \mathbb{Q}^s \mid PAx = 0\} = \{x \in \mathbb{Q}^s \mid P^{-1}PAx = P^{-1}0\} \\
&= \{x \in \mathbb{Q}^s \mid Ax = 0\} = \ker(A), \\
\ker(AQ) &= \{x \in \mathbb{Q}^s \mid AQx = 0\} = \{Q^{-1}y \in \mathbb{Q}^s \mid Ay = 0\} \\
&= Q^{-1}\{y \in \mathbb{Q}^s \mid Ay = 0\} = Q^{-1} \cdot \ker(A),
\end{aligned}$$

$$\begin{aligned}
\operatorname{im}(PA) &= \{PAx \mid x \in \mathbb{Q}^s\} = P\{Ax \mid x \in \mathbb{Q}^s\} = P \cdot \operatorname{im}(A), \\
\operatorname{im}(AQ) &= \{AQx \mid x \in \mathbb{Q}^s\} = \{Ay \mid Q^{-1}y \in \mathbb{Q}^s\} \\
&= \{Ay \mid y \in \mathbb{Q}^s\} = \operatorname{im}(A)
\end{aligned}$$

so that

$$\begin{aligned}
\ker(PA) &= \ker(A), & \ker(AQ) &= Q^{-1} \cdot \ker(A), \\
\operatorname{im}(PA) &= P \cdot \operatorname{im}(A), & \operatorname{im}(AQ) &= \operatorname{im}(A).
\end{aligned}$$

These relations specify how the kernel and image change if A is multiplied (on the left or the right) by an invertible matrix.

Let $t, s \in \mathbb{Z}_{>0}$ and let E_{ij} be the $t \times s$ matrix with 1 in the (i,j) entry and 0 elsewhere. Let $r \in \{1, \ldots, \min(s,t)\}$ and let

$$1_r = E_{11} + \cdots + E_{rr}.$$

Then

$$\ker(1_r) = \mathrm{span}\{e_{r+1}, \ldots, e_s\}.$$

For example, if $s = 5$ and $t = 6$ and $r = 2$ then

$$1_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \ker(1_2) = \mathrm{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Let $t, s \in \mathbb{Z}_{>0}$ and let E_{ij} be the $t \times s$ matrix with 1 in the (i, j) entry and 0 elsewhere. Let $r \in \{1, \ldots, \min(s, t)\}$ and let

$$1_r = E_{11} + \cdots + E_{rr}.$$

Let $e_1, \ldots, e_t$ be the standard basis of $\mathbb{Q}^t$. Then

$$\mathrm{im}(1_r) = \mathrm{span}\{e_{r+1}, \ldots, e_t\}.$$

For example, if $s = 5$ and $t = 6$ and $r = 2$ then

$$1_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathrm{im}(1_2) = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

Example V27&28. Let

$$S = \{|1,3,-1,1\rangle,\ |2,6,0,4\rangle,\ |3,9,-2,4\rangle\ \}.$$

Then

$$S = \left\{ \begin{pmatrix} 1 \\ 3 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 9 \\ -2 \\ 4 \end{pmatrix} \right\}$$

and

$$\mathbb{R}\text{-span}(S) = \text{colspace}(A) = \text{im}(A),\ \text{where } A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 6 & 0 \\ -1 & 0 & -2 \\ 1 & 4 & 4 \end{pmatrix}.$$

Now

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 3 & 6 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} = P 1_2 Q,$$

where

$$P = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 3 & 6 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{pmatrix}, \quad 1_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $\mathrm{colspace}(A) = \mathrm{im}(A)$ and

$$\mathrm{im}(A) = \mathrm{im}(P 1_2 Q) = P\mathrm{im}(1_2) = \mathrm{span}\left\{ P \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, P \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

Thus

$$\mathrm{colspace}(A) = \mathrm{im}(A) \quad \text{has basis} \quad \left\{ \begin{pmatrix} 1 \\ 3 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \\ 0 \\ 4 \end{pmatrix} \right\}.$$

Since

$$\ker(A) = \ker(P 1_2 Q) = \ker(1_2 Q) = Q^{-1} \ker(1_2)$$

and $\quad Q^{-1} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} \quad$ then $\quad \ker(A) = \mathbb{R}\text{-span}\left\{ \begin{pmatrix} -2 \\ \frac{1}{2} \\ 1 \end{pmatrix} \right\}.$

Thus $\ker(A)$ has basis $\{|-2, \frac{1}{2}, 1\rangle\}$.