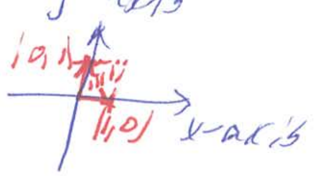
21.01.2026 (1)
Linear Algebra
Lect. 17 A. Ram

Vector space: $\mathbb{R}^2$

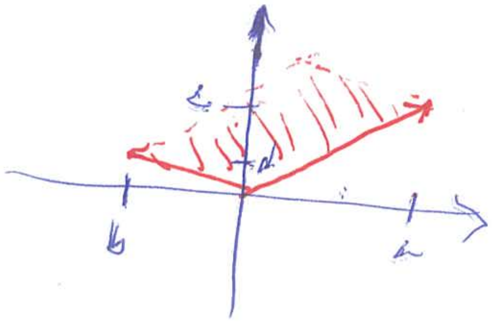


Basis of $\mathbb{R}^2$: $S = \{ |1,0\rangle, |0,1\rangle \}$

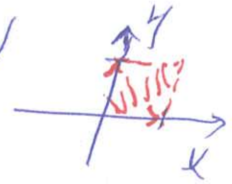
The linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$
with $T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}a\\c\end{pmatrix}$ and $T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}b\\d\end{pmatrix}$

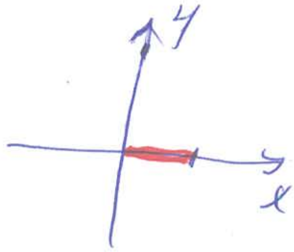
has matrix with respect to the basis S,

$$[T]_{SS} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$



If

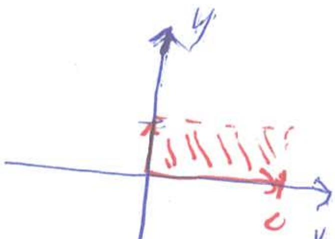
$[T]_{SS} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ then $T\Bigg($ 

$\Bigg) =$ 

and $T: \mathbb{R}^2 \to \mathbb{R}^2$ is projection onto the x-axis

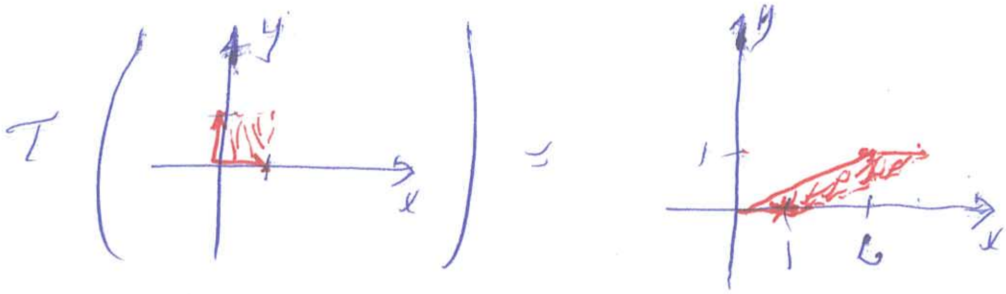
If

$[T]_{SS} = \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}$ then $T\Bigg($ 

$\Bigg) =$ 

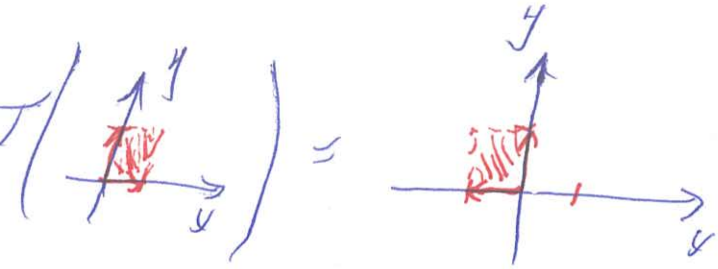
and $T: \mathbb{R}^2 \to \mathbb{R}^2$ stretches the x-axis by a factor of c.

If $[T]_{SS} = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$ then



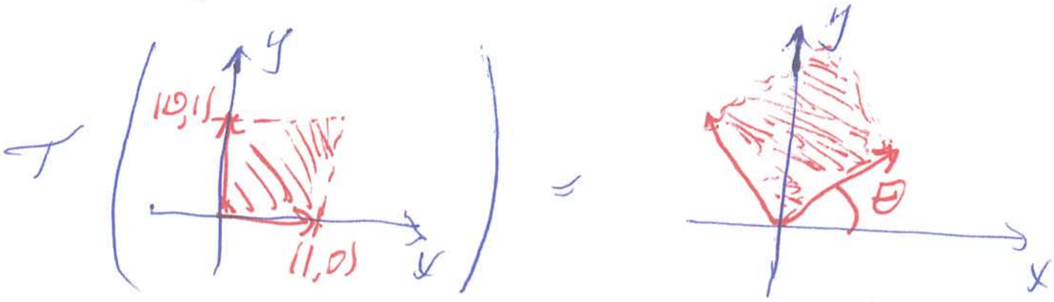
and T is a shear along the x-axis by a factor of c.

If $[T]_{SS} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ then



and T is reflection about the y-axis.

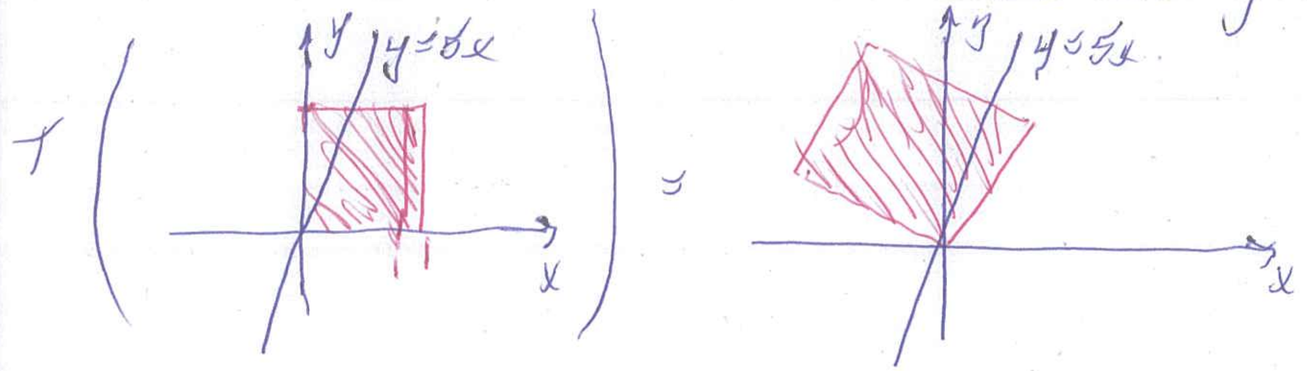
If $[T]_{SS} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ then



and T is a rotation by an angle θ about the origin $(0,0)$.

Example EV1 and LT6 and 23

Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ be reflection in the line $y=5x$.



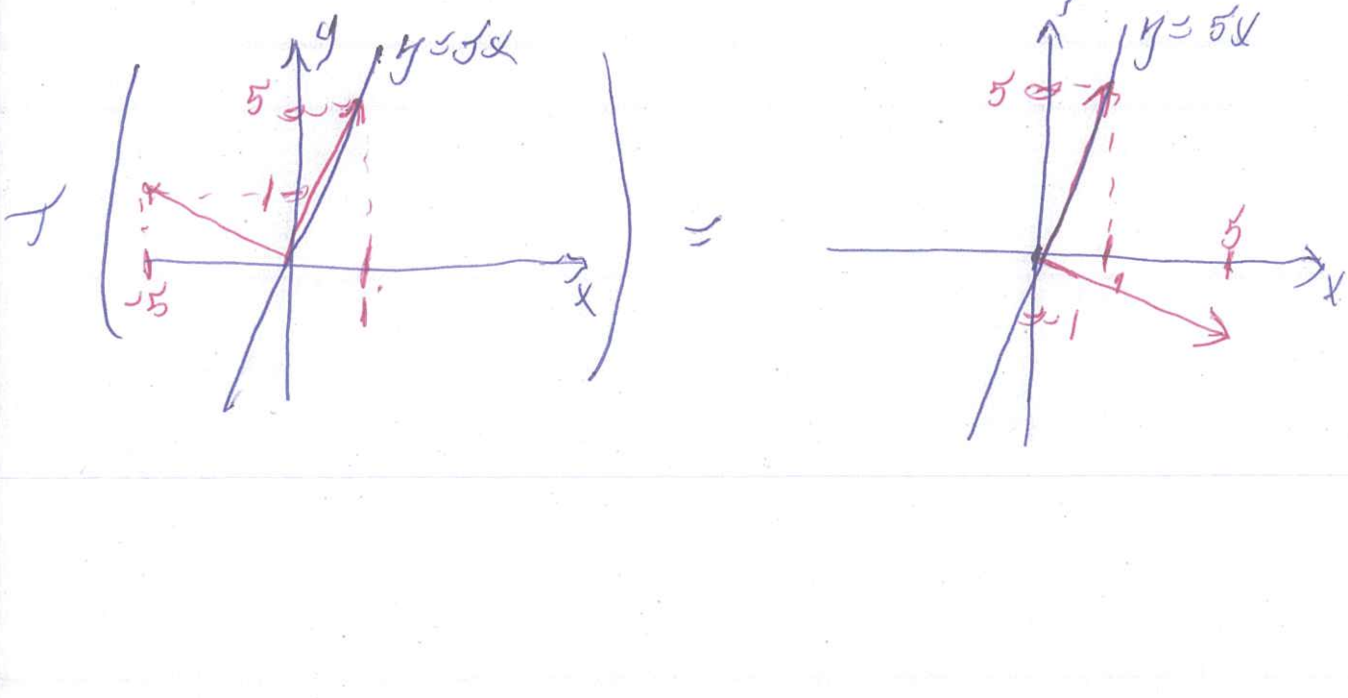
$S = \{|1,0\rangle, |0,1\rangle\}$ is the favorite basis of $\mathbb{R}^2$

$B = \{|1,5\rangle, |-5,1\rangle\}$ is another basis of $\mathbb{R}^2$

Then $T\begin{pmatrix}1\\5\end{pmatrix} = \begin{pmatrix}1\\5\end{pmatrix} = 1\cdot|1,5\rangle + 0\cdot|-5,1\rangle$

$T\begin{pmatrix}-5\\1\end{pmatrix} = \begin{pmatrix}5\\-1\end{pmatrix} = 0\cdot|1,5\rangle + (-1)\cdot|-5,1\rangle$

So $[T]_{BB} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$



The change of basis matrix from B to S is

$$[I]_{SB} = \begin{pmatrix} 1 & -5 \\ 5 & 1 \end{pmatrix}$$ since $|1,5\rangle = 1\cdot|1,0\rangle + 5\cdot|0,1\rangle$
$|-5,1\rangle = (-5)\cdot|1,0\rangle + 1\cdot|0,1\rangle$

Then

$$[I]_{BS} = \frac{1}{26}\begin{pmatrix} 1 & 5 \\ -5 & 1 \end{pmatrix} = [I]_{SB}^{-1}$$

So

$$[T]_{SS} = [I]_{SB}[T]_{BB}[I]_{BS}$$

$$= \begin{pmatrix} 1 & -5 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \frac{1}{26} & \frac{5}{26} \\ -\frac{5}{26} & \frac{1}{26} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 5 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} \frac{1}{26} & \frac{5}{26} \\ -\frac{5}{26} & \frac{1}{26} \end{pmatrix} = \begin{pmatrix} -\frac{24}{26} & \frac{10}{26} \\ \frac{10}{26} & \frac{24}{26} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{12}{13} & \frac{5}{13} \\ \frac{5}{13} & \frac{12}{13} \end{pmatrix}$$

If

$$A = \begin{pmatrix} -\frac{12}{13} & \frac{5}{13} \\ \frac{5}{13} & \frac{12}{13} \end{pmatrix}$$ and $$P = \begin{pmatrix} \frac{1}{26} & \frac{5}{26} \\ -\frac{5}{26} & \frac{1}{26} \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 1 & 5 \\ 5 & -1 \end{pmatrix}$$

and $$PAP^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$ which is diagonal.