26.01.2026 (1)
Linear Algebra
Lect 16 A. Rann

Coordinates

V a vector space

$B = \{b_1, b_2, \dots, b_n\}$ a basis of V.

$v \in V$.

The coordinate vector of v with respect to the basis B is

$$[v]_B = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \quad \text{if} \quad v = c_1 b_1 + c_2 b_2 + \cdots + c_n b_n$$

Example V34 Vector space: $\mathbb{Q}[x]_{\leq 2}$.

Bases of $\mathbb{Q}[x]_{\leq 2}$: $S = \{1, x, x^2\}$

$B = \{2, \frac{1}{2}x, -3x^2\}$

Vector: $p = 2 + 7x - 9x^2$.

$$[p]_S = \begin{pmatrix} 2 \\ 7 \\ 9 \end{pmatrix} \quad \text{since } 2 + 7x - 9x^2 = 2 \cdot 1 + 7x - 9x^2$$

$$[p]_B = \begin{pmatrix} 1 \\ 14 \\ 3 \end{pmatrix} \quad \text{since } 2 + 7x - 9x^2 = 1 \cdot 2 + 14 \cdot \tfrac{1}{2}x + 3 \cdot (-3x^2)$$

21.01.2026 ②
Linear Algebra
Lect. 16 A. Ram

<u>Example V33</u> Vectorspace: $\mathbb{R}^2$.

Bases of $\mathbb{R}^2$: $S = \{|1,0\rangle, |0,1\rangle\}$.

$\mathcal{B} = \{|2,1\rangle, |-1,1\rangle\}$.

Vector: $v = |1,5\rangle$.

$[v]_S = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$ since $|1,5\rangle = 1\cdot|1,0\rangle + 5\cdot|0,1\rangle$

$[v]_B = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ since $|1,5\rangle = 2\cdot|2,1\rangle + 3\,|-1,1\rangle$.

$T: V \to W$ a linear transformation.

$B = \{b_1, \ldots, b_s\}$ a basis of V.

$D = \{d_1, \ldots, d_t\}$ a basis of W.

The <u>matrix of T with respect to the basis B and the basis D</u> is

$$[T]_{DB} = \begin{pmatrix} a_{11} & \cdots & a_{1s} \\ a_{21} & \cdots & a_{2s} \\ \vdots & & \vdots \\ a_{t1} & \cdots & a_{ts} \end{pmatrix} \quad \text{if}$$

$T(b_1) = a_{11}d_1 + a_{21}d_2 + \cdots + a_{t1}d_t,$

$T(b_2) = a_{12}d_1 + a_{22}d_2 + \cdots + a_{t2}d_t,$

$\vdots$

$T(b_s) = a_{1s}d_1 + a_{2s}d_2 + \cdots + a_{ts}d_t.$

21.01.2026 (3)
Linear Algebra
Lect 14. A. Ram

Example LT 2 and 14

Linear transformation: $\mathbb{R}[x]_{\le 3} \to \mathbb{R}[x]_{\le 2}$ given by

$$T(a_0 + a_1 x + a_2 x^2 + a_3 x^3) = a_1 + 2a_2 x + 3a_3 x^2.$$

$S = \{1, x, x^2, x^3\}$ is a basis of $\mathbb{R}[x]_{\le 3}$.
$B = \{1, x, x^2\}$ is a basis of $\mathbb{R}[x]_{\le 2}$.

$$[T]_{BS} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \quad \text{since}$$

$T(1) = 0 = 0 + 0x + 0x^2$,
$T(x) = 1 = 1 + 0x + 0x^2$,
$T(x^2) = 2x = 0 + 2x + 0x^2$,
$T(x^3) = 3x^2 = 0 + 0x + 3x^2$.

Example LT 11 Linear transformation $T: M_{2\times2}(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$ given by

$T(Q) = Q^t$, where Q^t is the transpose of the matrix Q.

$S = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ is a basis of $M_{2\times2}(\mathbb{R})$.

$$[T]_{SS} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{since}$$

$T(E_{11}) = E_{11} = E_{11} + 0E_{12} + 0E_{21} + 0E_{22}$,

$T(E_{12}) = E_{21} = 0E_{11} + 0E_{12} + 1 \cdot E_{21} + 0E_{22}$,

$T(E_{21}) = E_{12} = 0E_{11} + 1 \cdot E_{12} + 0 \cdot E_{21} + 0E_{22}$,

$T(E_{22}) = E_{22} = 0E_{11} + 0E_{12} + 0E_{21} + 1 \cdot E_{22}$.

V a vector space.

$B = \{b_1, \dots, b_n\}$ a basis of V

$D = \{d_1, \dots, d_n\}$ another basis of V.

The <u>change of basis matrix from the basis B to the basis D</u> is

$$[I]_{DB} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & a_{2n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \quad \text{if}$$

$b_1 = a_{11}d_1 + a_{21}d_2 + \cdots + a_{n1}d_n$

$b_2 = a_{12}d_1 + a_{22}d_2 + \cdots + a_{n2}d_n$

$\vdots$

$b_n = a_{1n}d_1 + a_{2n}d_2 + \cdots + a_{nn}d_n$.

21.01.2026 (5)
Linear Algebra
Lect 16. A. Ravn.

Let

$T: U \to V$ and $S: V \to W$

be linear transformations. Let

$\mathcal{B} = \{u_1, \dots, u_k\}$ be a basis of U,

$\mathcal{C} = \{v_1, \dots, v_s\}$ a basis of V,

$\mathcal{D} = \{w_1, \dots, w_t\}$ a basis of W.

Then $(S \circ T): U \to W$ and

$$[S \circ T]_{\mathcal{D}\mathcal{B}} = [S]_{\mathcal{D}\mathcal{C}} [T]_{\mathcal{C}\mathcal{B}}.$$

Let V be a vector space and let

$I: V \to V$ given by $I(v) = v$,

be the <u>identity</u> transformation. Let

$\mathcal{B} = \{v_1, \dots, v_n\}$ be a basis of V,

$S = \{e_1, \dots, e_n\}$ be another basis of V.

Then

$$[I]_{\mathcal{B}\mathcal{B}} = 1, \quad [I]_{SS} = 1, \quad [I]_{S\mathcal{B}} [I]_{\mathcal{B}S} = [I]_{SS} = 1$$

and $[I]_{S\mathcal{B}} = [I]^{-1}_{\mathcal{B}S}$.