Lecture 16: With respect to a basis

Even in an arbitrary vector space, vectors and linear transformations can be converted to matrices, *provided that the corresponding column vectors and matrices are constructed with respect to a basis*.

Definition (Basis)

Let V be an $\mathbb{F}$-vector space. A *basis* of V is a set $S = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ such that every vector in V is a unique linear combination of $\mathbf{b}_1, \ldots, \mathbf{b}_n$.

Definition (Coordinates)

Let $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be an ordered basis for an $\mathbb{F}$-vector space V and let $v \in V$. The *coordinate vector of v with respect to $\mathcal{B}$* is $[v]_B \in \mathbb{F}^n$ given by

$$[v]_B = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \quad \text{if} \quad v = c_1\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n.$$

Example V33. The coordinate vector of $v = (1,5)$ with respect to the basis $S = \{(1,0),(0,1)\}$ of $\mathbb{R}^2$ is

$$[v]_S = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \qquad \text{since} \quad (1,5) = 1 \cdot (1,0) + 5 \cdot (0.1).$$

The coordinate vector of $v = (1,5)$ with respect to the basis $B = \{(2,1),(-1,1)\}$ of $\mathbb{R}^2$ is

$$[v]_B = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \qquad \text{since} \quad (1,5) = 2 \cdot (2,1) + 3 \cdot (-1.1).$$

Example V34. The coordinate vector of $p = 2 + 7x - 9x^2$ with respect to the basis $B = \{2, \frac{1}{2}x, -3x^2\}$ of $\mathbb{Q}[x]_{\leqslant 2}$ is

$$[p]_B = \begin{pmatrix} 1 \\ 14 \\ 3 \end{pmatrix} \quad \text{since} \quad 2 + 7x - 9x^2 = 1 \cdot 2 + 14 \cdot (\tfrac{1}{2}x) + 3 \cdot (-3x^2).$$

Definition

Let $f\colon V \to W$ be an $\mathbb{F}$-linear transformation. Let $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_s\}$ be a basis of V and let $C = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_t\}$ be a basis of W. Suppose that

$$\begin{aligned}
f(\mathbf{b}_1) &= A_{11}\mathbf{c}_1 + A_{21}\mathbf{c}_2 + \cdots + A_{n1}\mathbf{c}_n,\\
f(\mathbf{b}_2) &= A_{12}\mathbf{c}_1 + A_{22}\mathbf{c}_2 + \cdots + A_{n2}\mathbf{c}_n,\\
&\vdots\\
f(\mathbf{b}_n) &= A_{1n}\mathbf{c}_1 + A_{2n}\mathbf{c}_2 + \cdots + A_{nn}\mathbf{c}_n,
\end{aligned}$$

The *matrix of f with respect to bases B and C* is the matrix

$$[f]_{CB} = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}$$

Definition (Change of basis matrix)

Let V be and $\mathbb{F}$-vector space. Let $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis of V and let $C = \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ be another basis of V. Let

$$\begin{aligned}
\mathbf{b}_1 &= A_{11}\mathbf{c}_1 + A_{21}\mathbf{c}_2 + \cdots + A_{n1}\mathbf{c}_n,\\
\mathbf{b}_2 &= A_{12}\mathbf{c}_1 + A_{22}\mathbf{c}_2 + \cdots + A_{n2}\mathbf{c}_n,\\
&\vdots\\
\mathbf{b}_n &= A_{1n}\mathbf{c}_1 + A_{2n}\mathbf{c}_2 + \cdots + A_{nn}\mathbf{c}_n,
\end{aligned}$$

The *change of basis matrix from B to C* is

$$[I]_{CB} = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}$$

The change of basis matrix is the matrix of the identity transformation I with respect to the basis B and C.

Let $T: U \to V$ and $f: V \to W$ be linear transformations. Let

B be a basis of U, C a basis of V D a basis of W.

Then

$$[f \circ T]_{DB} = [f]_{DC}[T]_{CB}.$$

Let $T: V \to W$ be a linear transformation.

S be a basis of V. C be a basis of W,
B be another basis of V, D be another basis of W.

Then

$$[I]_{DC}[T]_{CB}[I]_{BS} = [T]_{DS} \quad \text{and} \quad [I]_{SB}[I]_{BS} = [I]_{SS} = 1.$$

This last equation tells us that $[I]_{SB}$ is invertible. Since invertible matrices must be square then B and S have the same number of elements.

Theorem

Let V be an $\mathbb{F}$-vector space. Any two bases of V have the same number of elements.

Example LT2&14. The derivative with respect to x is the linear transformation $T\colon \mathbb{R}[x]_{\leqslant 3} \to \mathbb{R}[x]_{\leqslant 2}$ given by

$$T(a_0 + a_1x + a_2x^2 + a_3x^3) = a_1 + 2a_2x + 3a_3x^2.$$

Since

$$\begin{aligned} T(1) &= T(1 + 0x + 0x^2 + 0x^3) = 0 + 0x + 0x^2, \\ T(x) &= T(0 + 1x + 0x^2 + 0x^3) = 1 + 0x + 0x^2, \\ T(x^2) &= T(0 + 0x + 1x^2 + 0x^3) = 0 + 2x + 0x^2, \\ T(x^3) &= T(0 + 0x + 0x^2 + 1x^3) = 0 + 0x + 3x^2, \end{aligned}$$

then the matrix of T with respect to the basis $S = \{1, x, x^2, x^3\}$ of $\mathbb{R}[x]_{\leqslant 3}$ and the basis $B = \{1, x, x^2\}$ of $\mathbb{R}[x]_{\leqslant 2}$ is

$$[T]_{BS} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Then

$$
\begin{aligned}
\ker(T) &= \{p \in \mathbb{R}[x]_{\leqslant 3} \mid T(p) = 0\} \\
&= \left\{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 \;\middle|\; \begin{array}{c} a_1 + 2a_2 x + 3a_3 x^2 \\ \quad = 0 + 0x + 0x^2 + 0x^3 \end{array} \right\} \\
&= \{a_0 + a_1 x + a_2 x^2 + a_3 x^3 \mid a_1 = 0 \text{ and } a_2 = 0 \text{ and } a_3 = 0\} \\
&= \{a_0 + 0x + 0x^2 + 0x^3 \mid a_0 \in \mathbb{R}\} \\
&= \{a_0 \mid a_0 \in \mathbb{R}\} = \mathbb{R}\text{-span}\{1\}
\end{aligned}
$$

and

$$
\begin{aligned}
\operatorname{im}(T) &= \{T(p) \mid p \in \mathbb{R}[x]_{\leqslant 3}\} \\
&= \{T(a_0 + a_1 x + a_2 x^2 + a_3 x^3) \mid a_0, a_1, a_2, a_3 \in \mathbb{R}\} \\
&= \{a_1 + 2a_2 x + 3a_3 x^2 \mid a_1, a_2, a_3 \in \mathbb{R}\} = \mathbb{R}[x]_{\leqslant 2},
\end{aligned}
$$

Example LT4. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ be the function given by

$$T(x_1, x_2, x_3) = |x_2 - 2x_3, 3x_1 + x_3\rangle = \begin{pmatrix} 0 & 1 & -2 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

With respect to the basis $S = \{|1, 0, 0\rangle, |0, 1, 0\rangle, |0, 0, 1\rangle\}$ of $\mathbb{R}^3$ and the basis $B = \{|1, 0\rangle, |0, 1\rangle\}$ of $\mathbb{R}^2$ the matrix of T is

$$[T]_{BS} = \begin{pmatrix} 0 & 1 & -2 \\ 3 & 0 & 1 \end{pmatrix}.$$

Example LT11. Let $T\colon M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be the linear transformation given by

$$T(Q) = Q^t.$$

Find the matrix of T with respect to the basis $B = \{E_{11}, E_{12}, E_{21}, E_{22}\}$, where E_{ij} is the matrix with 1 in the (i,j) entry and 0 elsewhere.
Since

$$\begin{aligned}
T(E_{11}) &= E_{11} = 1\cdot E_{11} + 0\cdot E_{12} + 0\cdot E_{21} + 0\cdot E_{22},\\
T(E_{12}) &= E_{21} = 0\cdot E_{11} + 0\cdot E_{12} + 1\cdot E_{21} + 0\cdot E_{22},\\
T(E_{21}) &= E_{12} = 0\cdot E_{11} + 1\cdot E_{12} + 0\cdot E_{21} + 0\cdot E_{22},\\
T(E_{22}) &= E_{12} = 0\cdot E_{11} + 0\cdot E_{12} + 0\cdot E_{21} + 1\cdot E_{22},
\end{aligned}$$

then the matrix of T with respect to the basis $B = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ is

$$[T]_{BB} = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Example LT12. Let $T\colon \mathbb{R}[x]_{\leqslant 2} \to \mathbb{R}[x]_{\leqslant 1}$ be the linear transformation given by

$$T(a_0 + a_1 x + a_2 x^2) = (a_0 + a_2) + a_0 x.$$

(a) Find the matrix of T with respect to the basis $B = \{1, x, x^2\}$ of $\mathbb{R}[x]_{\leqslant 2}$ and the basis $C = \{1, x\}$ of $\mathbb{R}[x]_{\leqslant 1}$.

(b) Find the matrix of T with respect to the basis $B = \{1, x, x^2\}$ of $\mathbb{R}[x]_{\leqslant 2}$ and the basis $D = \{2, 3x\}$ of $\mathbb{R}[x]_{\leqslant 1}$.

Let $b_1 = 1$, $b_2 = x$, $b_3 = x^2$ and $c_1 = 2$, $c_2 = 3x$ and $d_1 = 1$, $d_2 = x$. Since

$$\begin{aligned} T(1) &= 1 + x & &= 1\cdot 1 + 1\cdot x & &= \tfrac{1}{2}\cdot 2 + \tfrac{1}{3}\cdot(3x), \\ T(x) &= 0 & &= 0\cdot 1 + 0\cdot x & &= 0\cdot 2 + 0\cdot(3x), \\ T(x^2) &= 1 & &= 1\cdot 1 + 0\cdot x & &= \tfrac{1}{2}\cdot 2 + 0\cdot(3x), \end{aligned}$$

then

$$\begin{aligned} T(b_1) &= 1\cdot d_1 + 1\cdot d_2, \\ T(b_2) &= 0\cdot d_1 + 0\cdot d_2, \\ T(b_3) &= 1\cdot d_1 + 0\cdot d_2, \end{aligned} \qquad \text{and} \qquad \begin{aligned} T(b_1) &= \tfrac{1}{2}c_1 + \tfrac{1}{3}c_2, \\ T(b_2) &= 0\cdot c_1 + 0\cdot c_2, \\ T(b_3) &= \tfrac{1}{2}\cdot c_1 + 0\cdot c_2, \end{aligned}$$

and

$$[T]_{DB} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad [T]_{CB} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 \end{pmatrix}.$$

Example LT13. Suppose that $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation and that the matrix of T with respect to the basis $A = \{|1,0,0\rangle, |0,1,0\rangle, |0,0,1\rangle\}$ of $\mathbb{R}^3$ and the basis $S = \{|1,0\rangle, |0,1\rangle\}$ of $\mathbb{R}^2$ is

$$[T]_{SA} = \begin{pmatrix} 5 & 1 & 0 \\ 1 & 5 & -2 \end{pmatrix}.$$

Find the matrix of T with respect to the basis $B = \{|1,1,0\rangle, |1,-1,0\rangle, |1,-1,-2\rangle\}$ of $\mathbb{R}^3$ and the basis $C = \{|1,1\rangle, |1,-1\rangle\}$ of $\mathbb{R}^2$.

The answer is

$$[T]_{CB} = \begin{pmatrix} 6 & 0 & 2 \\ 0 & 4 & 2 \end{pmatrix}$$

since

$$T(1,1,0) = \begin{pmatrix} 5 & 1 & 0 \\ 1 & 5 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \end{pmatrix} = 6 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 0 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

$$T(1,-1,0) = \begin{pmatrix} 5 & 1 & 0 \\ 1 & 5 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \end{pmatrix} = 0 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 4 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

$$T(1,-1,-2) = \begin{pmatrix} 5 & 1 & 0 \\ 1 & 5 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$