Lecture 15: Kernel and image of a linear transformation

Definition (Kernel and image of a linear transformation)

The *kernel* of an $\mathbb{F}$-linear transformation $f\colon V \to W$ is the set

$$\ker(f) = \{v \in V \mid f(v) = 0\}.$$

The *image* of an $\mathbb{F}$-linear transformation $f\colon V \to W$ is the set

$$\operatorname{im}(f) = \{f(v) \mid v \in V\}.$$

Definition (Kernel and image of a matrix)

Let $A \in M_{t\times s}(\mathbb{Q})$. The *kernel of A* is

$$\ker(A) = \{x \in \mathbb{Q}^s \mid Ax = 0\}$$

and the *image of A* is

$$\operatorname{im}(A) = \{Ax \mid s \in \mathbb{Q}^s\}.$$

Example A5. Let $T : V \to W$ be an $\mathbb{R}$-linear transformation. Show that $\ker(T) = \{v \in V \mid T(v) = 0\}$ is a subspace of V.

Let $v_1, v_2 \in \ker(T)$. Then

$$T(v_1 + v_2) = T(v_1) + T(v_2) = 0 + 0 = 0. \quad \text{So } v_1 + v_2 \in \ker(T).$$

Subtracting $T(0)$ from each side of the equation $T(0) = T(0 + 0) = T(0) + T(0)$ gives

$$0 = T(0), \quad \text{and so} \quad 0 \in \ker(T).$$

Let $v \in \ker(T)$ and let $c \in \mathbb{R}$. Then

$$T(cv) = cT(v) = c \cdot 0 = 0 \quad \text{and so} \quad cv \in \ker(T).$$

So $\ker(T)$ is a subspace of V.

Example A6. Let $T\colon V \to W$ be an $\mathbb{R}$-linear transformation. Show that $\operatorname{im}(T) = \{T(v) \mid v \in V\}$ is a subspace of W.
Subtracting $T(0)$ from each side of the equation $T(0) = T(0+0) = T(0) + T(0)$ gives

$$0 = T(0), \quad \text{and so} \quad 0 \in \operatorname{im}(T).$$

Let $w_1, w_2 \in W$. Then there exist $v_1, v_2 \in V$ such that

$$T(v_1) = w_1 \quad \text{and} \quad T(v_2) = w_2.$$

Then $w_1 + w_2 = T(v_1) + T(v_2) = T(v_1 + v_2)$,

$$\text{and so} \quad w_1 + w_2 \in \operatorname{im}(T).$$

Let $w \in W$ and let $c \in \mathbb{R}$. Then there exists $v \in V$ such that

$$T(v) = w.$$

Then $cw = cT(v) = T(cv)$

$$\text{and so} \quad cw \in \operatorname{im}(T).$$

So $\operatorname{im}(T)$ is a subspace of W.

Definition (Injective, surjective, bijective, invertible)

Let S and T be sets and let $f\colon S \to T$ be a function from S to T.

(a) The function $f\colon S \to T$ is *injective* if f satistifes

$$\text{if } s_1, s_2 \in S \text{ and } f(s_1) = f(s_2) \quad \text{then } s_1 = s_2.$$

(b) The function $f\colon S \to T$ is *surjective* if f satisfies

$$\text{if } t \in T \quad \text{then there exists } s \in S \text{ such that } f(s) = t.$$

(c) The function $f\colon S \to T$ is *bijective* if f is

both injective and surjective.

(d) The function $f\colon S \to T$ is *invertible* if there exists a function $g\colon T \to S$ such that

$$g \circ f = \mathrm{Id}_S \quad \text{and} \quad f \circ g = \mathrm{Id}_T\,.$$

Definition

Let V be a vector space. The *dimension* of V is

$$\dim(V) = (\text{number of elements in a basis } B \text{ of } V).$$

Theorem (The rank-nullity theorem)

Let $f\colon V \to W$ be an $\mathbb{F}$-linear transformation. Then

(a) $\ker(f)$ *is a subspace of* V.

(b) $\mathrm{im}(f)$ *is a subspace of* W.

(c) $\dim(\ker(f) + \dim(im(f)) = \dim(V)$.

Theorem

Let $f\colon V \to W$ be an $\mathbb{F}$-linear transformation. Then

(a) *f is injective if and only if* $\ker(f) = \{0\}$.

(b) *f is surjective if and only if $im(f) = W$.*

(c) *f is invertible if and only if f is both injective and surjective.*

Example LT15. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by

$$T(x, y, z) = (2x - y, y + z).$$

Find bases for $\ker(T)$ and $\mathrm{Im}(T)$ and verify the rank-nullity theorem.

$$\begin{aligned}
\ker(T) &= \{|x, y, z\rangle \in \mathbb{R}^3 \mid T(x, y, z) = |0, 0\rangle\} \\
&= \{|x, y, z\rangle \in \mathbb{R}^3 \mid |2x - y, y + z\rangle = |0, 0\rangle\} \\
&= \left\{ |x, y, z\rangle \in \mathbb{R}^3 \;\middle|\; \begin{array}{r} 2x - y = 0, \\ y + z = 0 \end{array} \right\} \\
&= \left\{ |x, y, z\rangle \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} x = \frac{1}{2}y, \\ y = y, \\ z = -y \end{array} \right\} \\
&= \{y \cdot |\tfrac{1}{2}, 1, 1\rangle \in \mathbb{R}^3 \mid y \in \mathbb{R}\} = \mathbb{R}\text{-span}\{|\tfrac{1}{2}, 1, -1\rangle\}
\end{aligned}$$

and $\{|\frac{1}{2}, 1, -1\rangle\}$ is a basis of $\ker(T)$. So $\dim(\ker(T)) = 1$.

Since

$$T(\tfrac{1}{2},0,0) = |1,0\rangle \quad \text{and} \quad T(0,0,1) = |0,1\rangle$$

then

$$|1,0\rangle \text{ and } |0,1\rangle \text{ are elements of } \mathrm{im}(T).$$

Since $\mathrm{im}(T)$ is a subspace of $\mathbb{R}^2$ then $\mathbb{R}\text{-span}\{|1,0\rangle, |0,1\rangle\}$ is a subset of $\mathrm{im}(T)$. So

$$\mathrm{im}(T) = \mathbb{R}^2 \quad \text{and} \quad \{|1,0\rangle, |0,1\rangle\} \text{ is a basis of } \mathrm{im}(T).$$

So $\dim(\mathrm{im}(T)) = 2$ and

$$\dim(\ker(T)) + \dim(\mathrm{im}(T)) = 2 + 1 = 3 \quad \text{and} \quad 3 = \dim(\mathbb{R}^3)$$

is the dimension of the source of the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^2$.

Example LT16&17. Let $T\colon \mathbb{R}[x]_{\leqslant 2} \to \mathbb{R}[x]_{\leqslant 1}$ be the linear transformation given by

$$T(a_0 + a_1x + a_2x^2) = (a_0 - a_1 + a_2)(1 + 2x).$$

(a) Find bases for $\ker(T)$ and $\operatorname{Im}(T)$.

(b) Is T injective?

(c) Is T surjective?

$$\begin{aligned}
\ker(T) &= \{a_0 + a_1x + a_2x^2 \in \mathbb{R}[x]_{\leqslant 2} \mid T(a_0 + a_1x + a_2x^2) = 0 + 0x\} \\
&= \{a_0 + a_1x + a_2x^2 \in \mathbb{R}[x]_{\leqslant 2} \mid (a_0 - a_1 + a_2)(1 + 2x) = 0 + 0x\} \\
&= \left\{ a_0 + a_1x + a_2x^2 \in \mathbb{R}[x]_{\leqslant 2} \;\middle|\; \begin{array}{r} a_0 - a_1 + a_2 = 0, \\ 2(a_0 - a_1 + a_2) = 0 \end{array} \right\} \\
&= \{a_0 + a_1x + a_2x^2 \in \mathbb{R}[x]_{\leqslant 2} \mid a_0 = a_1 - a_2\} \\
&= \{(a_1 - a_2) + a_1x + a_2x^2 \mid a_1, a_2 \in \mathbb{R}\} \\
&= \{a_1(1 + x) + a_2(-1 + x^2) \mid a_1, a_2 \in \mathbb{R}\} \\
&= \mathbb{R}\text{-span}\{1 + x, -1 + x^2\}
\end{aligned}$$

and $\{1+x, -1+x^2\}$ is a basis of $\ker(T)$.

$$\begin{aligned}\operatorname{im}(T) &= \{\, T(a_0 + a_1 x + a_2 x^2) \mid a_0, a_1, a_2 \in \mathbb{R}\} \\ &= \{(a_0 - a_1 + a_2)(1+2x) \mid a_0, a_1, a_2 \in \mathbb{R}\} \\ &= \{a(1+2x) \mid a \in \mathbb{R}\} = \mathbb{R}\text{-span}\{1+2x\}\end{aligned}$$

and $\{1+2x\}$ is a basis of $\operatorname{im}(T)$. So $\dim(\operatorname{im}(T)) = 1$.

Since $\ker(T) \neq \{0\}$ then T is not injective.

Since $\mathbb{R}[x]_{\leqslant 1} = \{c_0 + c_1 x \mid c_1, c_2 \in \mathbb{R}\}$ then

$$\operatorname{im}(T) \neq \mathbb{R}[x]_{\leqslant 1} \quad \text{and} \quad T \text{ is not surjective.}$$