Lecture 14: span, linear independence and bases

Definition (Basis and dimension)

Let $\mathbb{F}$ be a field and let V be an $\mathbb{F}$-vector space.
Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a subset of V.
An *$\mathbb{F}$-linear combination* of $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is an element of the set

$$\mathbb{F}\text{-span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \{c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k \mid c_1, c_2, \ldots, c_k \in \mathbb{F}\}.$$

The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is *linearly independent over $\mathbb{F}$* if it satisfies:

$$\text{if } c_1, \ldots, c_k \in \mathbb{F} \text{ and } c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = 0$$

$$\text{then } c_1 = 0,\ c_2 = 0,\ \ldots,\ c_k = 0.$$

An *$\mathbb{F}$-basis* of V is a subset $B \subseteq V$ such that

(a) $\mathbb{F}\text{-span}(B) = V$,

(b) B is linearly independent.

The *$\mathbb{F}$-dimension* of V is the number of elements of a $\mathbb{F}$-basis B of V.

Example A9. Let V be a $\mathbb{Q}$-vector space and let $v_1, v_2, v_3, v_4, v_5 \in V$. Let $S = \{v_1, v_2, v_3, v_4, v_5\}$. Show that $\mathbb{Q}$-span(S) is a subspace of V.

(a) Since $0 = 0v_1 + 0v_2 + 0v_3 + 0v_4 + 0v_5$ then $0 \in \mathbb{Q}$-span(S).

(b) Assume $a = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 + a_5v_5 \in \mathbb{Q}$-span$(S)$ and $b = b_1v_1 + b_2v_2 + b_3v_3 + b_4v_4 + b_5v_5 \in \mathbb{Q}$-span$(S)$. Then

$$\begin{aligned} a + b &= a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 + a_5v_5 \\ &\quad + b_1v_1 + b_2v_2 + b_3v_3 + b_4v_4 + b_5v_5 \\ &= (a_1 + b_1)v_1 + (a_2 + b_2)v_2 + (a_3 + b_3)v_3 \\ &\quad + (a_4 + b_4)v_4 + (a_5 + b_5)v_5. \end{aligned}$$

So $a + b \in \mathbb{Q}$-span(S).

(c) Assume $a = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 + a_5v_5 \in \mathbb{Q}$-span$(S)$ and $c \in \mathbb{Q}$. Then

$$\begin{aligned} ca &= c(a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 + a_5v_5) \\ &= ca_1v_1 + ca_2v_2 + ca_3v_3 + ca_4v_4 + ca_5v_5 \in \mathbb{Q}\text{-span}(S). \end{aligned}$$

So $\mathbb{Q}$-span(S) is a subspace of V.

Example V13. In $\mathbb{R}[x]_{\leqslant 2}$, is $1 - 2x - x^2 \in \mathbb{R}\text{-span}\{1 + x + x^2, 3 + x^2\}$?

By definition $\mathbb{R}\text{-span}\{1 + x + x^2, 3 + x^2\}$
$$= \{c_1(1 + x + x^2) + c_2(3 + x^2) \mid c_1 c_2 \in \mathbb{R}\}.$$
So we need to show that there exist $c_1, c_2 \in \mathbb{R}$ such that

$$c_1(1 + x + x^2) + c_2(3 + x^2) = 1 - 2x - x^2.$$

So we need to show that the system $\begin{aligned} c_1 + 3c_2 &= 1, \\ c_1 + 0c_2 &= -2, \\ c_1 + c_2 &= -1, \end{aligned}$ has a solution.

The second equation gives $c_1 = -2$ and then $c_2 = -1 - c_1 = 1 + 2 = 3$.
Since the equation $c_1 + 3c_2 = 1$ also works when $c_1 = -2$ and $c_2 = 3$ then $c_1 = -2$, $c_2 = 1$ is a solution to this system.

Alternatively, the solution can be found by row reduction as follows. In matrix form the equations are

$$\begin{pmatrix} 1 & 3 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & 3 \\ 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & 1 \\ 0 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}.$$

So $c_1 = -2$ and $c_2 = 1$ is a solution.

So $-2(1 + x + x^2) + (3 + x^2) = 1 - 2x - x^2$.

So $1 - 2x - x^2 \in \mathbb{R}\text{-span}\{1 + x + x^2, 3 + x^2\}$.

So $1 - 2x - x^2$ is a linear combination of $1 + x + x^2$ and $3 + x^2$ and

$$1 - 2x - x^2 \in \mathbb{R}\text{-span}\{1 + x + x^2, 3 + x^2\}.$$

$\square$

Example V14. Let S be the subset of $\mathbb{R}^3$ given by

$$S = \{(1,1,1),(2,2,2),(3,3,3)\}. \qquad \text{Determine } \mathbb{R}\text{-span}(S).$$

In this case

$$\begin{aligned}
\mathbb{R}\text{-span}(S) &= \{c_1 |1,1,1\rangle + c_2 |2,2,2\rangle + c_3 |3,3,3\rangle \mid c_1, c_2, c_3 \in \mathbb{R}\} \\
&= \{c_1 |1,1,1\rangle + 2c_2 |1,1,1\rangle + 3c_3 |1,1,1\rangle \mid c_1, c_2, c_3 \in \mathbb{R}\} \\
&= \{(c_1 + 2c_2 + 3c_3) |1,1,1\rangle \mid c_1, c_2, c_3 \in \mathbb{R}\} \\
&= \{t |1,1,1\rangle \mid t \in \mathbb{R}\} \\
&= \{|t,t,t\rangle \mid t \in \mathbb{R}\}.
\end{aligned}$$

Here $\{\ |1,1,1\rangle\ \}$ is a basis of $\mathbb{R}$-span(S) and

$$\dim(\mathbb{R}\text{-span}(S)) = 1 \quad \text{(even though } S \text{ has 3 elements).}$$

□

Example V16new. Let S be the subset of $\mathbb{R}[x]_{\leqslant 2}$ given by

$S = \{1+2x, 1+5x+3x^2, x+x^2\}$. Show that $\text{span}(S) = \mathbb{R}[x]_{\leqslant 2}$.

Proof. By definition

$$\mathbb{R}\text{-span}(S) = \{c_1(1+2x)+c_2(1+5x+3x^2)+c_3(x+x^2) \mid c_1, c_2, c_3 \in \mathbb{R}\}.$$

To show: (a) $\mathbb{R}\text{-span}(S) \subseteq \mathbb{R}^3$
(b) $\mathbb{R}^3 \subseteq \mathbb{R}\text{-span}(S)$.

(a) Since $S \subseteq \mathbb{R}^3$ and $\mathbb{R}^3$ is closed under addition and scalar multiplication then $\mathbb{R}\text{-span}(S) \subseteq \mathbb{R}[x]_{\leqslant 2}$.

(b) To show: $\mathbb{R}[x]_{\leqslant 2} \subseteq \text{span}(S)$.

To show: $\mathbb{R}\text{-span}\{1, x, x^2\} \subseteq \text{span}(S)$.

Since $\mathbb{R}\text{-span}(S)$ is closed under addition and scalar multiplication,

To show: $\{1, x, x^2\} \subseteq \mathbb{R}\text{-span}(S)$.

To show: There exist $c_1, c_2, c_3, d_1, d_2, d_3, r_1, r_2, r_3 \in \mathbb{R}$ such that

$$\begin{aligned} c_1(1+2x) + c_2(1+5x+3x^2) + c_3(x+x^2) &= 1+0x+x^2, \\ d_1(1+2x) + d_2(1+5x+3x^2) + d_3(x+x^2) &= 0+x+0x^2, \\ r_1(1+2x) + r_2(1+5x+3x^2) + r_3(x+x^2) &= 0+0x+x^2. \end{aligned}$$

To show: There exist $c_1, c_2, c_3, d_1, d_2, d_3, r_1, r_2, r_3 \in \mathbb{R}$ such that

$$\begin{pmatrix} 1 & 1 & 0 \\ 2 & 5 & 1 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} c_1 & d_1 & r_1 \\ c_2 & d_2 & r_2 \\ c_3 & d_3 & r_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ to get } \begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} c_1 & d_1 & r_1 \\ c_2 & d_2 & r_2 \\ c_3 & d_3 & r_3 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \text{ to get } \begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 & d_1 & r_1 \\ c_2 & d_2 & r_2 \\ c_3 & d_3 & r_3 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

Since the bottom row on the left hand side is all 0 and the bottom row on the right hand sides is not all 0 then there *do not exist* $c_1, c_2, c_3, d_1, d_2, d_3, r_1, r_2, r_3 \in \mathbb{R}$ such that

$$\begin{pmatrix} 1 & 1 & 0 \\ 2 & 5 & 1 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} c_1 & d_1 & r_1 \\ c_2 & d_2 & r_2 \\ c_3 & d_3 & r_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

So $\{1, x, x^2\} \not\subseteq \mathbb{R}\text{-span}(S)$.

So $\text{span}(S) \neq \mathbb{R}[x]_{\leqslant 2}$. □

Example V23. Is $S = \{(1,-1),(2,4)\}$ a basis of $\mathbb{R}^2$?
Let

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}. \qquad \text{Then} \qquad A^{-1} = \frac{1}{6}\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{6} & \frac{1}{6} \end{pmatrix}.$$

So

$$\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \text{gives} \qquad \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

So S is linearly independent.

If $|a,b\rangle \in \mathbb{R}^2$ then $|a,b\rangle = c_1|1,-1\rangle + c_2|2,4\rangle$, where

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{6} & \frac{1}{6} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{2}{3}a - \frac{1}{3}b \\ \frac{1}{6}a + \frac{1}{6}b \end{pmatrix}.$$

So $\mathbb{R}^2 \subseteq \mathbb{R}\text{-span}(S)$. Since $S \subseteq \mathbb{R}^2$ and $\mathbb{R}^2$ is closed under addition and scalar multiplication then $\mathbb{R}\text{-span}(S) \subseteq \mathbb{R}^2$. So $\mathbb{R}\text{-span}(S) = \mathbb{R}^2$.

So S is a basis of $\mathbb{R}^2$.

Example V21. Let S be the subset of $M_2(\mathbb{R})$ given by

$$S = \left\{ \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 10 \\ 4 & 2 \end{pmatrix} \right\}.$$

Is S linearly independent?

To show: If $c_1, c_2, c_3 \in \mathbb{R}$ and

$$c_1 \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} + c_2 \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix} + c_3 \begin{pmatrix} 1 & 10 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

then $c_1 = 0$, $c_2 = 0$, $c_3 = 0$.

Suppose an oracle tells you to try (or you guess) $c_1 = -3$, $c_2 = 1$, $c_3 = -1$ and then you verify that

$$-3 \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 10 \\ 4 & 2 \end{pmatrix} = -3 \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} -3 & -9 \\ -3 & -3 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

This means that you don't have to have c_1, c_2 and c_3 all 0 to get a zero linear combination.
So S is not linearly independent.

If you have no oracle, or are not a good guesser, then proceed as follows. Assume $c_1, c_2, c_3 \in \mathbb{R}$ and

$$c_1 \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} + c_2 \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix} + c_3 \begin{pmatrix} 1 & 10 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then

$$\begin{aligned} c_1 - 2c_2 + c_3 &= 0, \\ 3c_1 + c_2 + 10c_3 &= 0, \\ c_1 + c_2 + 4c_3 &= 0, \\ c_1 - c_2 + 2c_3 &= 0, \end{aligned} \quad \text{or, equivalently,} \quad \begin{pmatrix} 1 & -2 & 1 \\ 3 & 1 & 10 \\ 1 & 1 & 4 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & -2 & 1 \\ 3 & 1 & 10 \\ 1 & -1 & 2 \\ 0 & 2 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & -2 & 1 \\ 1 & -1 & 2 \\ 0 & 4 & 4 \\ 0 & 2 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & -1 & 2 \\ 0 & -1 & -1 \\ 0 & 4 & 4 \\ 0 & 2 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & -1 & 2 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & -1 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This gives the system

$$\begin{aligned} c_1 + 3c_3 &= 0, \\ c_2 + c_3 &= 0, \end{aligned} \qquad \text{which is} \qquad \begin{aligned} c_1 &= -3c_3 \\ c_2 &= -c_3, \\ c_3 &= c_3, \end{aligned}$$

which has solutions

$$\left\{ c_3 \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix} \;\middle|\; c_3 \in \mathbb{R} \right\} = \mathbb{R}\text{-span} \left\{ \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

So $c_1 = 0$, $c_2 = 0$, $c_3 = 0$ is not the only solution.
This means that you don't have to have c_1, c_2 and c_3 all 0 to get a zero linear combination.
So S is not linearly independent.