# Lecture 13: Linear transformations

Linear transformations are for comparing vector spaces.

**Definition**

Let $\mathbb{F}$ be a field and let $V$ and $W$ be $\mathbb{F}$-vector spaces. An *$\mathbb{F}$-linear transformation from $V$ to $W$* is a function $f: V \to W$ such that

(a) If $v_1, v_2 \in V$ then $f(v_1 + v_2) = f(v_1) + f(v_2)$,

(b) If $c \in \mathbb{F}$ and $v \in V$ then $f(cv) = cf(v)$.

Example A1. Let $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix}$.

Let $T\colon \mathbb{R}^4 \to \mathbb{R}^2$ be the function given by $T(x) = Ax$ so that

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 + 3x_3 + 4x_4 \\ 5x_5 + 6x_6 + 7x_7 + 8x_8 \end{pmatrix}.$$

Show that $T$ is a linear transformation.

Let $u, v \in \mathbb{R}^4$. Then, by the distributive property of matrix multiplication,

$$T(u + v) = A(u + v) = Au + Av = T(u) + T(v).$$

Let $u \in R^4$ and $c \in \mathbb{R}$. Then, by the associative property of scalar multiplication for matrices,

$$T(cu) = Acu = cAu = cT(u).$$

So $T$ is a linear transformation.

Example A2. Let $t, s \in \mathbb{Z}_{>0}$ and $A \in M_{t \times s}(\mathbb{R})$. Let $T \colon \mathbb{R}^s \to \mathbb{R}^t$ be the function given by

$$T(x) = Ax.$$

Show that $T$ is a linear transformation.
Let $u, v \in \mathbb{R}^s$. Then, by the distributive property of matrix multiplication for matrices,

$$T(u+v) = A(u+v) = Au + Av = T(u) + T(v).$$

Let $u \in \mathbb{R}^s$ and $c \in \mathbb{R}$. Then, by the associative property of scalar multiplication for matrices,

$$T(cu) = Acu = cAu = cT(u).$$

Example A3. Let $n \in \mathbb{Z}_{>0}$ and let $T\colon M_n(\mathbb{Q}) \to \mathbb{Q}$ by the function given by

$$T(A) = \mathrm{Tr}(A).$$

Show that $T$ is a $\mathbb{Q}$-linear transformation.
Let $A, B \in M_{n\times n}(\mathbb{Q})$. Then

$$\begin{aligned} T(A+B) = \mathrm{Tr}(A+B) &= (A+B)_{11} + \cdots + (A+B)_{nn} \\ &= A_{11} + B_{11} + \cdots + A_{nn} + B_{nn} \\ &= A_{11} + \cdots + A_{nn} + +B_{11} + \cdots B_{nn} \\ &= \mathrm{Tr}(A) + \mathrm{Tr}(B) = T(A) + T(B). \end{aligned}$$

Let $A \in M_{n\times n}(\mathbb{Q})$ and $c \in \mathbb{Q}$. Then

$$\begin{aligned} T(cA) = \mathrm{Tr}(cA) &= (cA)_{11} + \cdots + (cA)_{nn} \\ &= cA_{11} + \cdots + cA_{nn} \\ &= c(A_{11} + \cdots + A_{nn}) = c\,\mathrm{Tr}(A) = cT(A). \end{aligned}$$

So $T$ is a linear transformation.

Example A4. Let $T\colon M_{2\times 2}(\mathbb{Q}) \to \mathbb{Q}$ be the function given by

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Show that $T$ is a linear transformation.
Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and let $c = 2$. Then

$$T(cA) = \det(2A) = \det\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 2\cdot 2 - 0\cdot 0 = 4$$

and

$$cT(A) = 2\det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2\cdot(1\cdot 1 - 0\cdot 0) = 2.$$

So this gives an example where $T(cA) \neq cT(A)$.
So $T$ *cannot possibly be a linear transformation*.