Vector spaces

$\mathbb{R}$-vector spaces have

addition and scalar multiplication by elements of $\mathbb{R}$

Examples

(1) $M_{2\times 3}(\mathbb{Q})$ is a $\mathbb{Q}$-vector space.

(2) The set of polynomials of degree ≤ 2 with complex coefficients

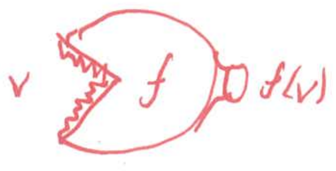
$$\{a_0 + a_1 x + a_2 x^2 \mid a_0, a_1, a_2 \in \mathbb{C}\} = \mathbb{C}[x]_{\le 2}$$

is a $\mathbb{C}$-vector space

$\mathbb{R}$-linear transformations are for comparing $\mathbb{R}$-vector spaces.

Let V and W be $\mathbb{R}$-vector spaces.

A $\mathbb{R}$-linear transformation from V to W is a function $f: V \to W$ such that



(a) If $v_1, v_2 \in V$ then

$$f(v_1 + v_2) = f(v_1) + f(v_2).$$ addition

(b) If $v \in V$ and $c \in \mathbb{R}$ then

$$f(cv) = cf(v).$$ scalar multiplication

Let V be an $\mathbb{R}$-vector space

A <u>subspace</u> of V is a subset $L \subseteq V$

$L \subseteq V$ such that

(a) $0 \in L$

(b) If $u, w \in L$ then $u + w \in L$

(c) If $u \in L$ and $c \in \mathbb{R}$ then $cu \in L$.

Let $S = \{v_1, v_2, \ldots, v_k\}$ be a subset of V.

$$\mathrm{span}(S) = \{c_1 v_1 + c_2 v_2 + \cdots + c_k v_k \mid c_1, c_2, \ldots, c_k \in \mathbb{R}\}$$

is the set of <u>linear combinations</u> of $v_1, \ldots, v_k$

The set S is <u>linearly independent</u> if S satisfies

if $c_1, c_2, \ldots, c_k \in \mathbb{R}$ and $c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0$

then $c_1 = 0, c_2 = 0, \ldots, c_k = 0$.

(In English: the <u>only</u> way to have a linear combination of $v_1, v_2, \ldots, v_k$ be 0 is to take all the scalar multiplications 0)

Example 21 Topic 4

Let $v_1, v_2, v_3, v_4 \in \mathbb{R}^3$ be given by

$v_1 = \langle 1,2,3\rangle$, $v_2 = \langle 3,6,9\rangle$, $v_3 = \langle -1,0,-2\rangle$, $v_4 = \langle 1,4,4\rangle$.

(a) Is $\{v_1, v_3\}$ linearly independent?

Assume $c_1 v_1 + c_2 v_2 = 0$. Then

$$c_1\langle 1,2,3\rangle + c_2\langle -1,0,-2\rangle = \langle 0,0,0\rangle.$$

and
$$\begin{aligned} c_1 - c_2 &= 0 \\ 2c_1 + 0c_2 &= 0 \\ 3c_1 - 2c_2 &= 0. \end{aligned}$$
so
$$\begin{aligned} c_1 &= c_2 \\ 2c_1 &= 0, \end{aligned}$$
so $c_1 = 0$, $c_2 = 0$.

So $\{v_1, v_3\}$ is linearly independent.

(b) Express v_2 and v_4 as linear combinations of v_1 and v_3.

Since
$$v_2 = \langle 3,6,9\rangle = 3\langle 1,2,3\rangle = 3v_1 = 3v_1 + 0v_3.$$
then v_2 is a linear combination of v_1 and v_3.

Is
$$v_4 = \langle 1,4,4\rangle = c_1\langle 1,2,3\rangle + c_2\langle -1,0,-2\rangle ?$$

Does
$$\begin{aligned} c_1 - c_2 &= 1 \\ 2c_1 + 0c_2 &= 4 \\ 3c_1 - 2c_2 &= 4 \end{aligned}$$
have a solution?

Since
$$\begin{aligned} 2 - 1 &= 1 \\ 2\cdot 2 + 0\cdot 1 &= 4 \\ 3\cdot 2 - 2\cdot 1 &= 4 \end{aligned}$$
then $c_1 = 2$, $c_2 = 1$ is a solution.

So $v_4 = 2v_1 + v_3$ and v_4 is a linear combination of v_1 and v_3.

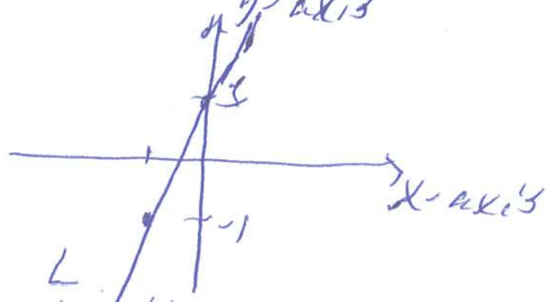
(c) Is $\{v_1, v_2, v_3, v_4\}$ linearly independent.

Since $2v_1 + 0v_2 + v_3 - v_4 = 0$ is a linear combination equal to 0 that is not the all 0 linear combination then $\{v_1, v_2, v_3, v_4\}$ is not linearly independent.

Example 7 Topic 4

Is $L = \{(x,y) \mid y = 2x+1\}$ a subspace of $\mathbb{R}^2$?

$$L = \left\{ \begin{pmatrix} x \\ 2x+1 \end{pmatrix} \,\middle|\, x \in \mathbb{R} \right\}$$



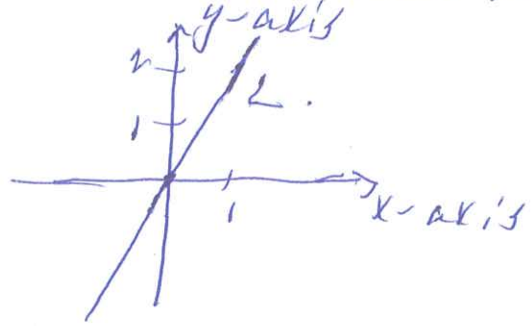
Since $\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0$ and $0 \neq 2\cdot 0 + 1$ then

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \notin L.$$

So L is not a subspace of $\mathbb{R}^2$.

Is $L=\{(x,y) \mid y=2x\}$ a subspace of $\mathbb{R}^2$?

$$L=\left\{\begin{pmatrix}x\\2x\end{pmatrix} \,\middle|\, x\in\mathbb{R}\right\}$$



(a) Since $0=\begin{pmatrix}0\\0\end{pmatrix}$ and $0=2\cdot 0$ then $0\in L$.

(b) Assume $\begin{pmatrix}a\\2a\end{pmatrix}\in L$ and $\begin{pmatrix}b\\2b\end{pmatrix}\in L$. Then

$$\begin{pmatrix}a\\2a\end{pmatrix}+\begin{pmatrix}b\\2b\end{pmatrix}=\begin{pmatrix}a+b\\2a+2b\end{pmatrix}=\begin{pmatrix}a+b\\2(a+b)\end{pmatrix}\in L.$$

(c) Assume $\begin{pmatrix}a\\2a\end{pmatrix}\in L$ and $c\in\mathbb{R}$. Then

$$c\begin{pmatrix}a\\2a\end{pmatrix}=\begin{pmatrix}ca\\c2a\end{pmatrix}=\begin{pmatrix}ca\\2\cdot(ca)\end{pmatrix}\in L.$$

So L is a subspace of $\mathbb{R}^2$.