Example (Topic 1 Ex 11) Find the values of $a, b \in \mathbb{R}$ for which the system

$$\begin{aligned} x-2y+z&=4\\ 2x-3y+z&=7\\ 3x-6y+az&=b \end{aligned}$$

has (a) no solution
(b) a unique solution
(c) LOTS of solutions.

In matrix form this system is

$$\begin{pmatrix} 3 & -6 & a\\ 2 & -3 & 1\\ 1 & -2 & 1 \end{pmatrix}\begin{pmatrix} x\\ y\\ z \end{pmatrix} = \begin{pmatrix} b\\ 7\\ 4 \end{pmatrix}$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & -2 \end{pmatrix} \text{ to get } \begin{pmatrix} 3 & -6 & a\\ 1 & -2 & 1\\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x\\ y\\ z \end{pmatrix} = \begin{pmatrix} b\\ 4\\ -1 \end{pmatrix}$$

Left multiply both sides by

$$\begin{pmatrix} 0 & 1 & 0\\ 1 & -3 & 0\\ 0 & 0 & 1 \end{pmatrix} \text{ to get } \begin{pmatrix} 1 & -2 & 1\\ 0 & 0 & a-3\\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x\\ y\\ z \end{pmatrix} = \begin{pmatrix} 4\\ b-12\\ -1 \end{pmatrix}$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0 \end{pmatrix} \text{ to get } \begin{pmatrix} 1 & -2 & 1\\ 0 & 1 & -1\\ 0 & 0 & a-3 \end{pmatrix}\begin{pmatrix} x\\ y\\ z \end{pmatrix} = \begin{pmatrix} 4\\ -1\\ b-12 \end{pmatrix}$$

Multiply both sides by

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ to get } \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & a-3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ b-12 \end{pmatrix}$$

<u>Case 1</u>: $a-3 \neq 0$. Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{a-3} \end{pmatrix} \text{ to get } \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ \frac{b-12}{a-3} \end{pmatrix}$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \text{ to get } \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 + \frac{b-12}{a-3} \\ \frac{b-12}{a-3} \end{pmatrix}$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ to get } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 + \frac{b-12}{a-3} \\ -1 + \frac{b-12}{a-3} \\ \frac{b-12}{a-3} \end{pmatrix}$$

So

$$x = 2 + \frac{b-12}{a-3}$$

$$y = -1 + \frac{b-12}{a-3}$$ is the unique solution.

$$z = \frac{b-12}{a-3}$$

Case 2: $a-3=0$. Then

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ b-12 \end{pmatrix}.$$

Case 2a: $b-12 \neq 0$.

If $a-3=0$ and $b-12\neq 0$ then the system has no solution.

Case 2b: $b-12=0$. If $a-3=0$ and $b-12=0$ then the system is

$$\begin{aligned} x-z&=2, \\ y-z&=-1, \end{aligned} \quad \text{which is} \quad \begin{aligned} x &= 2+z \\ y &= -1+z \\ z &= 0+z \end{aligned}$$

where z can be any number. So

$$Sol(A\vec{x}=\vec{b}) = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \mathrm{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$$

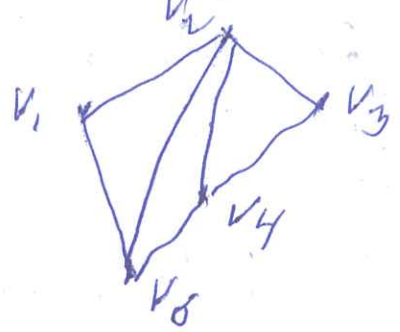
and there are LOTS of solutions.

Application to Graphs and networks

Square matrices with 0,1 entries are equivalent to graphs.

Example 1 and 2 (Topic 2 Examples 1 and 2)

The graph



has adjacency matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

There is a 1 in the (i,j)-entry if there is an edge connecting vertex i and vertex j.

$$A^3 = A \cdot A \cdot A = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 & 2 & 1 \\ 1 & 4 & 1 & 2 & 2 \\ 1 & 1 & 2 & 1 & 2 \\ 2 & 2 & 1 & 3 & 1 \\ 1 & 2 & 2 & 1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 4 & 3 & 3 & 5 \\ 6 & 6 & 6 & 7 & 7 \\ 3 & 6 & 2 & 5 & 3 \\ 3 & 7 & 5 & 4 & 7 \\ 5 & 7 & 3 & 7 & 4 \end{pmatrix}$$

the (i,j) entry of A^3 gives the number of paths of length 3 from vertex i to vertex j.

Example 13 Topic 3 Find a vector form Lect 11 A. Ram for the line of intersection of the two planes

$$x+3y+2z=6 \quad \text{and} \quad 3x+2y+z=11.$$

The points on the intersection are the (x,y,z) such that

$$\begin{matrix} 3x+2y+z=1 \\ x+3y+2z=6 \end{matrix} \quad \text{which is} \quad \begin{pmatrix} 3 & 2 & 1 \\ 1 & 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 11 \\ 6 \end{pmatrix}$$

Left multiply both sides by

$$\begin{pmatrix} 0 & 1 \\ 1 & -3 \end{pmatrix} \text{ to get } \begin{pmatrix} 1 & 3 & 2 \\ 0 & -7 & -7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -7 \end{pmatrix}$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 \\ 0 & \frac{-1}{7} \end{pmatrix} \text{ to get } \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$$

Left multiply both sides by

$$\begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix} \text{ to get } \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

So

$$\begin{matrix} x-z=3 \\ y+z=1 \end{matrix} \quad \text{and} \quad \begin{matrix} x=3+z \\ y=1-z \\ z=0+z. \end{matrix}$$

So the solutions are

$$Sol(Ax=b) = \left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \;\middle|\; z \in \mathbb{R} \right\}$$

which is a line through $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ in direction $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.