MAST10007 Linear Algebra

THE UNIVERSITY OF MELBOURNE
SCHOOL OF MATHEMATICS AND STATISTICS

Summer Term, 2026

Arun Ram: Additional Slides

These slides have been made by Arun Ram, in preparation for teaching of the summer session of MAST10007 Linear Algebra at University of Melbourne in 2026. The template is from the University of Melbourne School of Mathematics and Statistics slide deck which was produced by members of the School including, in particular, huge developments by Craig Hodgson and Christine Mangelsdorf.

Solving problems with an unknown parameter.

Example L11. Find the values of $a, b \in \mathbb{Q}$ for which the system

$$\begin{aligned} x - 2y + z &= 4, \\ 2x - 3y + z &= 7, \\ 3x - 6y + az &= b, \end{aligned} \quad \text{has} \quad \begin{array}{l} \text{(a) no solution,} \\ \text{(b) a unique solution,} \\ \text{(c) LOTS of solutions.} \end{array}$$

In matrix form this system is

$$\begin{pmatrix} 3 & -6 & a \\ 2 & -3 & 1 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b \\ 7 \\ 4 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 3 & -6 & a \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b \\ 4 \\ -1 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & -2 & 1 \\ 0 & 0 & a-3 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ b-12 \\ -1 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & a-3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ b-12 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & a-3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ b-12 \end{pmatrix}.$$

Case 1: $a-3 \neq 0$. Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{a-3} \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ \frac{b-12}{a-3} \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 + \frac{b-12}{a-3} \\ \frac{b-12}{a-3} \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 + \frac{b-12}{a-3} \\ -1 + \frac{b-12}{a-3} \\ \frac{b-12}{a-3} \end{pmatrix}.$$

So

$$
\begin{aligned}
x &= 2 + \tfrac{b-12}{a-3}, \\
y &= -1 + \tfrac{b-12}{a-3}, \\
z &= \tfrac{b-12}{a-3},
\end{aligned}
$$

or, equivalently,

$$\mathrm{Sol}(A\mathbf{x} = \mathbf{b}) = \left\{ \begin{pmatrix} 2 + \frac{b-12}{a-3} \\ -1 + \frac{b-12}{a-3} \\ \frac{b-12}{a-3} \end{pmatrix} \right\} \qquad \text{(exactly } \textit{one} \text{ solution).}$$

Case 2: $a - 3 = 0$. Then

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ b - 12 \end{pmatrix}.$$

Case 2a: $b - 12 \neq 0$. If $a - 3 = 0$ and $b - 12 \neq 0$ then this system has *no solution*.

Case 2b: $b - 12 = 0$. If $a - 3 = 0$ and $b - 12 = 0$ then this system is

$$\begin{aligned} x - z &= 2, \\ y - z &= -1, \end{aligned} \qquad \text{which is} \qquad \begin{aligned} x &= 2 + z, \\ y &= -1 + z, \\ z &= 0 + z, \end{aligned}$$

where z can be any number. So

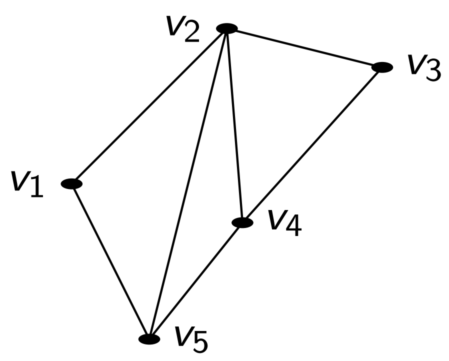
$$\mathrm{Sol}(A\mathbf{x} = \mathbf{b}) = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\},$$

and there are *LOTS of solutions*.

Application to graphs and networks.

Square matrices with 0, 1 entries are equivalent to graphs.

Example M1&2. The graph



has adjacency matrix $A = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}$

There is a 1 in the (i, j) entry if there is an edge connecting vertex i and vertex j.

Then

$$A^3 = A(A^2) = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 & 2 & 1 \\ 1 & 4 & 1 & 2 & 2 \\ 1 & 1 & 2 & 1 & 2 \\ 2 & 2 & 1 & 3 & 1 \\ 1 & 2 & 2 & 1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 6 & 3 & 3 & 5 \\ 6 & 6 & 6 & 7 & 7 \\ 3 & 6 & 2 & 5 & 3 \\ 3 & 7 & 5 & 4 & 7 \\ 5 & 7 & 3 & 7 & 4 \end{pmatrix}.$$

The (i, j) entry of A^3 gives the number of paths of length three from vertex i to vertex j.

Definition (Transpose of a matrix)

Let $A \in M_{t\times s}(\mathbb{Q})$. The *transpose of A* is $A^T \in M_{s\times t}(\mathbb{Q})$ given by

$$(A^T)_{ij} = A_{ji}, \quad \text{for } i \in \{1, \dots, s\} \text{ and } j \in \{1, \dots, t\}.$$

Example M4. If $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ then $A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$.

Definition (Trace)

Let $A \in M_{n\times n}(\mathbb{Q})$. The *trace of A* is

$$\mathrm{Tr}(A) = A_{11} + \cdots + A_{nn} \quad \text{where} \quad A = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix}$$

Example M3.

$$\mathrm{Tr}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = 1 + 5 + 9 = 15.$$

Theorem (Inverse of a 2×2 matrix)

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{Q})$. *Then*

1. *If* $ad - bc \neq 0$ *then* $A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
2. *If* $ad - bc = 0$ *then* A^{-1} *does not exist.*

Example M5. Let $A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$. Then

$$A^{-1} = \frac{1}{(2\cdot 1 - 1\cdot(-1))}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{-1}{3} & \frac{2}{3} \end{pmatrix}.$$

Check:

$$\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{-1}{3} & \frac{2}{3} \end{pmatrix} = \begin{pmatrix} \frac{3}{3} & 0 \\ 0 & \frac{3}{3} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$