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Linear Algebra
Lect 10
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<u>Cross products are only in $\mathbb{R}^3$</u>

$\mathbb{R}^3 = M_{3\times 1}(\mathbb{R})$. Let $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$

The <u>cross product of $\vec{v}$ and $\vec{w}$</u> is

$$\vec{v} \times \vec{w} = \begin{pmatrix} +(v_2 w_3 - w_3 w_2) \\ -(v_1 w_3 - w_3 v_1) \\ +(v_1 w_2 - w_2 v_1 \end{pmatrix}$$

<u>Theorem</u>

(a) $\langle \vec{u}, \vec{v}\times\vec{w} \rangle = \det \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}$

(b) $\langle \vec{v}, \vec{v}\times\vec{w} \rangle = 0$ and $\langle \vec{w} (\vec{v}\times\vec{w} \rangle = 0$.

(c) $|\langle \vec{u}, \vec{v}\times\vec{w} \rangle| = \begin{pmatrix} \text{volume of parallelepiped} \\ \text{with edges } \vec{u}, \vec{v}, \vec{w} \end{pmatrix}$

<u>Dirac notation:</u> $|u_1, u_2, u_3\rangle = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$

$\langle u_1, u_2, u_3| = (u_1 \ u_2 \ u_3)$

$\hat{\imath} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\hat{\jmath} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\hat{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

<u>Note:</u> Part (b) says that $\vec{v}\times\vec{w}$ is perpendicular to both $\vec{v}$ and $\vec{w}$.

Example 5 Topic 3

Find a vector perpendicular to both $\langle 1,1,1\rangle$ and $\langle 1,-1,-2\rangle$.

The vector

$$\vec{n} = \langle 1,1,1\rangle \times \langle 1,-1,-2\rangle = \begin{pmatrix} +(1\cdot(-2) - 1\cdot(-1)) \\ -(1\cdot(-2) - 1\cdot 1) \\ +(1\cdot(-1) - 1\cdot 1) \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix}$$

is perpendicular to both $\langle 1,1,1\rangle$ and $\langle 1,-1,-2\rangle$.

Check:

$$\langle 1,1,1 \mid -1,3,-2\rangle = 1\cdot(-1) + 1\cdot 3 + 1\cdot(-2) = -1+3-2 = 0.$$

$$\langle 1,-1,-2 \mid -1,3,-2\rangle = 1\cdot(-1) + (-1)\cdot 3 + (-2)\cdot(-2) = -1-3+4 = 0.$$

Example 10 Topic 3

Find the Cartesian equation of the plane with vector equation

$$\{\langle x,y,z\rangle = s\langle 1,-1,0\rangle + t\langle 2,0,1\rangle + \langle -1,1,1\rangle, \text{ with } s,t\in\mathbb{R}\}$$

Let $\vec{u} = \langle 1,-1,0\rangle$ and $\vec{v} = \langle 2,0,1\rangle$.

A normal vector to the plane is

$$\vec{n} = \vec{u}\times\vec{v} = \langle 1,-1,0\rangle \times \langle 2,0,1\rangle = \begin{pmatrix} +(-1\cdot 1 - 0\cdot 0) \\ -(1\cdot 1 - 0\cdot 2) \\ +(1\cdot 0 - (-1)\cdot 2) \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$$

Then $\langle -1,1,1\rangle$ is a point on the plane and

$$\langle \vec{n} \mid -1,1,1\rangle = \langle -1,-1,2 \mid -1,1,1\rangle = 1-1+2 = 2.$$

The plane is the set

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; \langle x,y,z \mid \vec{n} \rangle = 2 \right\}$$

$$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; \langle x,y,z \mid -1,-1,2 \rangle = 2 \right\}$$

$$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; -x-y+2z = 2 \right\}.$$

<u>Example 7 Topic 3</u> Find the volume of the parallelipiped with adjacent edges $\vec{PQ}, \vec{PR}, \vec{PS}$, where

$P = \langle 2,0,-1 \rangle$, $Q = \langle 4,1,0 \rangle$, $R = \langle 3,-1,1 \rangle$, $S = \langle 2,-2,2 \rangle$

The edges of the parallelipiped are

$\vec{PQ} = Q - P = \langle 2,1,1 \rangle$

$\vec{PR} = R - P = \langle 1,-1,2 \rangle$

$\vec{PS} = S - P = \langle 0,-2,3 \rangle$

then

$$\begin{pmatrix} \text{volume of} \\ \text{parallelipiped} \end{pmatrix} = \left| \langle \vec{PQ} \mid \vec{PR} \times \vec{PS} \rangle \right|$$

$$= \left| \det \begin{pmatrix} 2 & 1 & 1 \\ 1 & -1 & 2 \\ 0 & -2 & 3 \end{pmatrix} \right|$$

$$= \left| \begin{matrix} 2 \cdot \det\begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} \\ -1 \cdot \det\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} + 1 \cdot \det\begin{pmatrix} 1 & -1 \\ 0 & -2 \end{pmatrix} \end{matrix} \right|$$

$$= \left| 2((-1)\cdot 3 - 2\cdot(-2)) - 1\cdot(1\cdot 3 - 2\cdot 0) + 1\cdot(1\cdot(-2) - (-1)\cdot 0) \right|$$

$$= |2\cdot 1 - 1\cdot 3 + 1\cdot(-2)| = |2-3-2| = |-3| = 3.$$

<u>Example 6 Topic 3</u> Find the area of the triangle in $\mathbb{R}^3$ with vertices

$\langle 2,-5,4\rangle$, $\langle 3,-4,5\rangle$ and $\langle 3,-6,2\rangle$.

Letting $\vec{u} = \langle 3,-4,5\rangle - \langle 2,-5,4\rangle = \langle 1,1,1\rangle$

and $\vec{v} = \langle 3,-6,2\rangle - \langle 2,-5,4\rangle = \langle 1,-1,-2\rangle$

then

$$\vec{u}\times\vec{v} = \langle 1,1,1\rangle \times \langle 1,-1,-2\rangle = \begin{pmatrix} +(1\cdot(-2) - 1\cdot(-1)) \\ -(1\cdot(-2) - 1\cdot 1) \\ +(1\cdot(-1) - 1\cdot 1) \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix}$$

Then

$$\begin{pmatrix}\text{area of} \\ \text{triangle}\end{pmatrix} = \frac{\frac{1}{2}\cdot 1}{\|\vec{u}\times\vec{v}\|} \begin{pmatrix}\text{volume of parallipiped with} \\ \text{edges } \vec{u}, \vec{v}, \text{ and } \vec{u}\times\vec{v}\end{pmatrix}$$

$$= \frac{\frac{1}{2}}{\|\vec{u}\times\vec{v}\|} \left| \langle \vec{u}\times\vec{v} \mid \vec{u}\times\vec{v}\rangle \right| = \frac{1}{2}\,\frac{\|\vec{u}\times\vec{v}\|^2}{\|\vec{u}\times\vec{v}\|}$$

$$= \frac{1}{2}\|\vec{u}\times\vec{v}\| = \frac{1}{2}\sqrt{(-1)^2+3^2+(-2)^2} = \frac{1}{2}\sqrt{1+9+4} = \frac{\sqrt{14}}{2}.$$