MAST10007 Linear Algebra

THE UNIVERSITY OF MELBOURNE
SCHOOL OF MATHEMATICS AND STATISTICS

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Arun Ram: Additional Slides

These slides have been made by Arun Ram, in preparation for teaching of the summer session of MAST10007 Linear Algebra at University of Melbourne in 2026. The template is from the University of Melbourne School of Mathematics and Statistics slide deck which was produced by members of the School including, in particular, huge developments by Craig Hodgson and Christine Mangelsdorf.

Lecture 10: Cross products (are only available in $\mathbb{R}^3$)

Let $\mathbf{i}, \mathbf{j}, \mathbf{k} \in \mathbb{R}^3$ be given by

$$\mathbf{i} = |1, 0, 0\rangle, \quad \mathbf{j} = |0, 1, 0\rangle, \quad \mathbf{k} = |0, 0, 1\rangle.$$

Proposition (Standard basis of $\mathbb{R}^3$)

Let $\mathbf{v} \in \mathbb{R}^3$.

(a) *If* $\mathbf{v} = |a_1, a_2, a_3\rangle$ *then* $\mathbf{v} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$.

(b) *If* $a_1, a_2, a_3 \in \mathbb{R}$ *and* $a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = 0$ *then* $a_1 = 0$ *and* $a_2 = 0$ *and* $a_3 = 0$.

Definition (Cross product)

Let $\mathbf{u} = |u_1, u_2, u_3\rangle \in \mathbb{R}^3$ and let $\mathbf{v} = |v_1, v_2, v_3\rangle \in \mathbb{R}^3$. The *cross product* of $\mathbf{u}$ and $\mathbf{v}$ is given by

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2)\mathbf{i} + (u_3 v_1 - u_1 v_3)\mathbf{j} + (u_1 v_2 - u_2 v_1)\mathbf{k}.$$

Definition (Cross product)

Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle \in \mathbb{R}^3$ and let $\mathbf{v} = \langle v_1, v_2, v_3 \rangle \in \mathbb{R}^3$. The *cross product* of $\mathbf{u}$ and $\mathbf{v}$ is given by

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2)\mathbf{i} + (u_3 v_1 - u_1 v_3)\mathbf{j} + (u_1 v_2 - u_2 v_1)\mathbf{k}.$$

In terms of determinants $\mathbf{u} \times \mathbf{v}$ is

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \det \begin{pmatrix} u_2 & u_3 \\ v_2 & v_3 \end{pmatrix} \mathbf{i} - \det \begin{pmatrix} u_1 & u_3 \\ v_1 & v_3 \end{pmatrix} \mathbf{j} + \det \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \mathbf{k} \\ &\text{``=''} \ \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}, \end{aligned}$$

where the last 3×3 determinant on the right hand side doesn't really make sense (because $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are not numbers); but this "determinant" is a very useful mnemonic.

If $\mathbf{u} = |u_1, u_2, u_3\rangle, \mathbf{v} = |v_1, v_2, v_3\rangle, \mathbf{w} = |w_1, w_2, w_3\rangle$ then

$$\begin{aligned}
\langle \mathbf{u}, \mathbf{v} \times \mathbf{w} \rangle &= \langle u_1, u_2, u_3 | (v_2 w_3 - w_3 v_2, -(v_1 w_3 - v_3 w_1), v_1 w_2 - v_2 w_1 \rangle \\
&= u_1(v_2 w_3 - v_3 w_2) - u_2(v_1 w_3 - v_3 w_1) + u_3(v_1 w_2 - v_2 w_1) \\
&= \det \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}.
\end{aligned}$$

Since

$$\langle \mathbf{v}, \mathbf{v} \times \mathbf{w} \rangle = \det \begin{pmatrix} v_1 & v_2 & v_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} = 0$$

and

$$\langle \mathbf{w}, \mathbf{v} \times \mathbf{w} \rangle = \det \begin{pmatrix} w_1 & w_2 & w_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} = 0$$

then

$\mathbf{v} \times \mathbf{w}$ is perpendicular to both $\mathbf{v}$ and $\mathbf{w}$.

Example E5. Find a vector perpendicular to both $|1, 1, 1\rangle$ and $|1, -1, -2\rangle$.

Solution: By definition of the cross product

$$
\begin{aligned}
&|1,1,1\rangle \times |1,-1,-2\rangle \\
&\quad = |1\cdot(-2) - 1\cdot(-1), -(1\cdot(-2) - 1\cdot 1), 1\cdot(-1) - 1\cdot 1\rangle \\
&\quad = |-1, 3, -2\rangle.
\end{aligned}
$$

The vector $|-1, 3, -2\rangle$ is perpendicular to both $|1, 1, 1\rangle$ and $|1, -1, -2\rangle$ since

$$\langle -1, 3, -2 \mid 1, 1, 1\rangle = -1 + 3 - 2 = 0$$

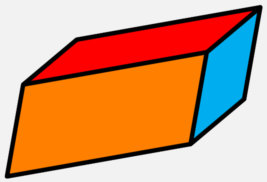
and

$$\langle -1, 3, -2 \mid 1, -1, -2\rangle = -1 - 3 + 4 = 0.$$

Theorem (Volumes of parallelipipeds)

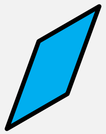
(3) *Let* $\mathbf{u} = |u_1, u_2, u_3\rangle \in \mathbb{R}^3$ *and* $\mathbf{v} = |v_1, v_2, v_3\rangle \in \mathbb{R}^3$ *and* $\mathbf{w} = |w_1, w_2, w_3\rangle \in \mathbb{R}^3$. *The volume of the parallelipiped with vertices* $0, \mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{w}, \mathbf{v}+\mathbf{w}, \mathbf{u}+\mathbf{v}+\mathbf{w}$ *is*

$$\left| \det \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} \right|.$$



(2) *Let* $\mathbf{u} = |u_1, u_2\rangle \in \mathbb{R}^2$ *and* $\mathbf{v} = |v_1, v_2\rangle \in \mathbb{R}^2$. *The area of the paralellogram with vertices* $0, \mathbf{u}, \mathbf{v}, \mathbf{u}+\mathbf{v}$ *is*

$$\left| \det \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \right|.$$



(1) *Let* $\mathbf{u} = |u_1\rangle \in \mathbb{R}^1$. *The length of the segment with endpoints* 0 *to* $\mathbf{u}$ *is*

$$|\det(u_1)|.$$

Example E6. Find the area of the triangle in $\mathbb{R}^3$ with vertices $|2,-5,4\rangle$, $|3,-4,5\rangle$ and $|3,-6,2\rangle$.

$$\text{Letting}\quad \mathbf{u} = |3,-4,5\rangle - |2,-5,4\rangle = |1,1,1\rangle \qquad \text{and}$$
$$\mathbf{v} = |3,-6,2\rangle - |2,-5,4\rangle = |1,-1,-2\rangle, \quad \text{then}$$

$$\begin{aligned}
\mathbf{u}\times\mathbf{v} &= |1,1,1\rangle \times |1,-1,-2\rangle \\
&= |1\cdot(-2) - 1\cdot(-1), -(1\cdot(-2) - 1\cdot 1), 1\cdot(-1) - 1\cdot 1\rangle \\
&= |-1,3,-2\rangle.
\end{aligned}$$

Then

$$\begin{aligned}
(\text{Area of triangle}) &= \tfrac{1}{2}(\text{area of rectangle with edges } \mathbf{u} \text{ and } \mathbf{v}) \\
&= \tfrac{1}{2}\frac{1}{\|\mathbf{u}\times\mathbf{v}\|}\begin{pmatrix} \text{volume of parallelipiped} \\ \text{with edges } \mathbf{u},\ \mathbf{v} \text{ and } \mathbf{u}\times\mathbf{v}) \end{pmatrix} \\
&= \tfrac{1}{2}\|\mathbf{u}\times\mathbf{v}\| = \tfrac{1}{2}\|\,|-1,3,-2\rangle\,\| \\
&= \tfrac{1}{2}\sqrt{(-1)^2 + 3^2 + (-2)^2} = \frac{\sqrt{14}}{2}.
\end{aligned}$$

Example E7. Find the volume of the parallelipiped with adjacent edges $\overrightarrow{PQ}$, $\overrightarrow{PR}$, $\overrightarrow{PS}$, where

$$P = |2,0,-1\rangle, \;\; Q = |4,1,0\rangle, \;\; R = |3,-1,1\rangle \text{ and } S = |2,-2,2\rangle.$$

Since the edges of the parallelipiped are

$$\overrightarrow{PQ} = P - Q = |2,1,1\rangle, \quad \overrightarrow{PR} = P - R = |1,-1,2\rangle,$$

$$\overrightarrow{PS} = P - S = |0,-2,3\rangle,$$

then

$$\begin{aligned}(\text{Volume of parallelipiped}) &= |\langle \overrightarrow{PQ}, \overrightarrow{PR} \times \overrightarrow{PS}\rangle| \\ &= \left|\det\begin{pmatrix}2 & 1 & 1\\ 1 & -1 & 2\\ 0 & -2 & 3\end{pmatrix}\right| = \left|2\cdot\det\begin{pmatrix}-1 & 2\\ -2 & 3\end{pmatrix} - \det\begin{pmatrix}1 & 1\\ -2 & 3\end{pmatrix}\right| \\ &= |2(-3+4) - (3+2)| = |-3| = 3.\end{aligned}$$

Example E10. Find the Cartesian equation of the plane with vector form

$$|x, y, z\rangle = s\,|1, -1, 0\rangle + t\,|2, 0, 1\rangle + |-1, 1, 1\rangle, \text{ with } s, t \in \mathbb{R}.$$

A normal vector to this plane is

$$\mathbf{n} = \mathbf{u} \times \mathbf{v}, \qquad \text{where } \mathbf{u} = |1, -1, 0\rangle \text{ and } \mathbf{v} = |2, 0, 1\rangle.$$

Then $\mathbf{u} \times \mathbf{v} = |-1-0, -(1-0), 0-(-2)\rangle = |-1, -1, 2\rangle$.
Then $|-1, 1, 1\rangle$ is a point in the plane, and

$$\langle -1, 1, 1 \,|\, \mathbf{u} \times \mathbf{v}\rangle = \langle -1, 1, 1 \,|\, -1, -1, 2\rangle = 1 - 1 + 2 = 2.$$

Since the plane is

$$|-1, 1, 1\rangle + \{|x, y, z\rangle \in \mathbb{R}^3 \mid \langle x, y, z \,|\, -1, -1, 2\rangle = 0\}$$

then the Cartesian equation of the plane is

$$-x - y + 2z = 2.$$