2.3 Kernels, images and solutions of linear systems

2.3.1 Kernels and images

Let $\mathbb{F}$ be a field. For $s \in \mathbb{Z}_{>0}$ let

$$\mathbb{F}^s = M_{s\times 1}(\mathbb{F}).$$

A $\mathbb{F}$-*subspace of* $\mathbb{F}^s$ is a subset $V \subseteq \mathbb{F}^s$ such that

(a) If $v_1, v_2 \in V$ then $v_1 + v_2 \in V$,
(b) if $v \in V$ and $c \in \mathbb{F}$ then $cv \in V$.

If $r \in \mathbb{Z}_{>0}$ and $v_1, \dots, v_r \in \mathbb{F}^s$ then

$$\mathbb{F}\text{-span}\{v_1, \dots, v_r\} = \{c_1v_1 + \cdots c_rv_r \mid c_1, \dots, c_r \in \mathbb{F}\}$$

is the smallest subspace of $\mathbb{F}^s$ containing $\{v_1, \dots, v_r\}$.

Let $A \in M_{t\times s}(\mathbb{F})$. Define

$$\ker(A) = \{v \in \mathbb{F}^s \mid Av = 0\} \qquad \text{and} \qquad \mathrm{im}(A) = \{Av \mid v \in \mathbb{F}^s\}.$$

The following proposition determines $\ker(A)$ in terms of its normal form decomposition $A = P1_rQ$.

Proposition 2.8. *Let $t, s \in \mathbb{Z}_{>0}$. Let $r \in \{1, \dots, \min(s,t)\}$ and set*

$$1_r = E_{11} + \cdots + E_{rr} \quad \textit{in } M_{t\times s}(\mathbb{F}).$$

Let $P \in GL_t(\mathbb{F})$ and $Q \in GL_s(\mathbb{F})$ and let $A = P1_rQ$. Then

$$\ker(A) = \ker(P1_rQ) = \ker(1_Q) = Q^{-1}\ker(1_r) = \mathbb{F}\text{-}\textit{span}\{\textit{last } s{-}r \textit{ columns of } Q^{-1}\}, \quad \textit{and}$$
$$\mathrm{im}(A) = \mathrm{im}(P1_rQ) = \mathrm{im}(P1_r) = P\mathrm{im}(1_r) = \mathbb{F}\text{-}\textit{span}\{\textit{first } r \textit{ columns of } P\}.$$

The following corollary is sometimes termed 'the rank-nullity theorem'.

Corollary 2.9. *Let $A \in M_{t\times s}(\mathbb{F})$.*

(a) $\ker(A)$ *is a subspace of* $\mathbb{F}^s$.

(b) $\mathrm{im}(A)$ *is a subspace of* $\mathbb{F}^t$.

(c) $\dim(\ker(A)) + \dim(\mathrm{im}(A)) = (\textit{number of columns of } A)$.

2.3.2 Solutions of systems of linear equations

Using matrix multiplication the system of equations

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1s}x_s &= b_1,\\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2s}x_s &= b_2,\\
&\vdots\\
a_{t1}x_1 + a_{m2}x_2 + \cdots + a_{ts}x_s &= b_t,
\end{aligned}$$

is written in the form

$$Ax = b, \qquad \text{where} \quad A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1s}\\ a_{21} & a_{22} & \cdots & a_{2s}\\ \vdots & & & \vdots\\ a_{t1} & a_{t2} & \cdots & a_{ts}\end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} b_1\\ b_2\\ \vdots\\ b_t\end{pmatrix}.$$

Define

$$\mathrm{Sol}(Ax=b) = \{x \in \mathbb{F}^s \mid Ax = b\}.$$

The following proposition says that if A is square and invertible then

$$x = A^{-1}Ax = A^{-1}b \qquad \text{is the unique solution to the system of equations } Ax=b,$$

so that $\mathrm{Sol}(Ax=b)$ contains only one element.

Proposition 2.10. *Let $A \in M_{t\times s}(\mathbb{F})$ and $b \in \mathbb{F}^t$. If $t = s$ and $A \in GL_t(\mathbb{F})$ then*

$$\mathrm{Sol}(Ax=b) = \{A^{-1}b\}.$$

The following proposition says that (if $\mathrm{Sol}(Ax=b)$ is nonempty then) $\mathrm{Sol}(Ax=b)$ is the same size as $\ker(A)$.

Proposition 2.11. *Let $A \in M_{t\times s}(\mathbb{F})$ and $b \in \mathbb{F}^t$ and assume $\mathrm{Sol}(Ax=b) \neq \emptyset$. Let $p \in \mathrm{Sol}(Ax=b)$. Then*

$$\mathrm{Sol}(Ax=b) = p + \ker(A).$$

The following proposition determines $\mathrm{Sol}(Ax=b)$ explicitly.

Proposition 2.12. *Let $A \in M_{t\times s}(\mathbb{F})$ and $b \in \mathbb{F}^t$. Assume $r \in \{1, \ldots, \min(s,t)\}$ and $P \in GL_t(\mathbb{F})$ and $Q \in GL_s(\mathbb{F})$ are such that*

$$A = P1_rQ.$$

Let $q_1, \ldots, q_s$ be the columns of Q^{-1}. Then

$$\mathrm{Sol}(Ax=b) = \begin{cases} Q^{-1}\begin{pmatrix} (P^{-1}b)_1 \\ \vdots \\ (P^{-1}b)_r \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \mathrm{span}\left\{ \begin{pmatrix} | \\ q_{r+1} \\ | \end{pmatrix}, \ldots, \begin{pmatrix} | \\ q_n \\ | \end{pmatrix} \right\}, & \textit{if the last } t-r \textit{ entries of } P^{-1}b \textit{ are equal to } 0, \\ \emptyset, & \textit{otherwise.} \end{cases}$$