2.2 Invertible matrices and normal form

2.2.1 Invertible matrices

A matrix $P \in M_n(\mathbb{F})$ is *invertible* if there exists a matrix $P^{-1} \in M_n(\mathbb{F})$ such that

$$PP^{-1} = 1 \qquad \text{and} \qquad P^{-1}P = 1.$$

The *general linear group* is

$$GL_n(\mathbb{F}) = \{P \in M_n(\mathbb{F}) \mid P \text{ is invertible}\}.$$

Proposition 2.5. *If $P, Q \in GL_n(\mathbb{F})$ then*

$$(PQ)^{-1} = Q^{-1}P^{-1}.$$

2.2.2 Normal form

Let $t, s \in \mathbb{Z}_{>0}$ and let $E_{ij} \in M_{t\times s}(\mathbb{F})$ denote the matrix with 1 in entry (i,j) and 0 elsewhere, For $r \in \{1, \ldots, \min(s,t)\}$ let

$$1_r \in M_{t\times s}(\mathbb{F}) \qquad \text{be given by} \qquad 1_r = E_{11} + \cdots + E_{rr}. \tag{1rdefn}$$

For example,

$$1_4 = \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix} \qquad \text{in } M_{7\times 15}(\mathbb{Q}).$$

The row reduction algorithm, or greedy normal form algorithm, (see Section ???) proves the following normal form theorem. There are more detailed statements of the output of the greedy normal form algorithm, but the following is what is needed to compute solutions of systems of linear equations.

Theorem 2.6. *Let $s, t \in \mathbb{Z}_{>0}$ and let E_{ij} denote the $t \times s$ matrix with 1 in the (i,j) entry and 0 elsewhere.*
Let $A \in M_{t\times s}(\mathbb{F})$. Then there exist

$$r \in \{1, \ldots, \min(s,t)\}, \qquad P \in GL_t(\mathbb{F}), \qquad Q \in GL_s(\mathbb{F})$$

$$\textit{such that} \qquad A = P1_rQ, \qquad \textit{where} \quad 1_r = E_{11} + \cdots + E_{rr}.$$

2.2.3 Orbit representatives for $GL_t(\mathbb{F})$ and $GL_s(\mathbb{F})$ acting on $M_{t\times s}(\mathbb{F})$

Let $s, t \in \mathbb{Z}_{>0}$. A *left orbit representative of* $M_{t\times s}(\mathbb{F})$ is $R \in M_{t\times s}(\mathbb{F})$ such that there exist $r \in \{0, 1, \ldots, \min(s,t)\}$ and $1 \le j_1 < \cdots < j_r \le s$ with

(a) If $j \in \{1, \ldots, r\}$ then

$$\begin{array}{rrrr} & & & R(1, j_i) = 0, \\ & & & R(2, j_i) = 0, \\ & & & \vdots \\ & & & R(j-1, j_i) = 0, \\ R(i,1) = 0, \quad R(i,2) = 0, & \ldots & R(i, j_i - 1) = 0, & R(i, j_i) = 1, \end{array}$$

(b) If $i \in \{r+1, \ldots, t\}$ then $R(i,1) = 0, R(i,2) = 0, \ldots, R(i,s) = 0$.

For example,

$$R = \begin{pmatrix} 0 & 0 & 1 & 8 & 3 & 0 & 9 & 1 & 3 & 4 & 0 & 2 & 6 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 1 & 5 & 2 & 2 & 8 & 0 & 4 & 1 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 6 & 3 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

is a left orbit representative in $M_{7\times 15}(\mathbb{Q})$. In this example, $t = 7$, $s = 15$, $r = 4$ and $i_1 = 3$, $i_2 = 6$, $i_3 = 11$, $i_4 = 14$.

Theorem 2.7. *Let $t, s \in \mathbb{Z}_{>0}$. Let $\mathcal{E}$ be the set of left orbit representatives of $M_{t\times s}(\mathbb{F})$. For $r \in \{1, \ldots, \min(t,s)\}$ let*

$$\text{let } 1_r \in M_{t\times s}(\mathbb{F}) \text{ be given by} \qquad 1_r = E_{11} + \cdots + E_{rr},$$

where E_{ij} has 1 in the (i,j) entry and 0 elsewhere. Let $1_0 = 0$ in $M_{t\times s}(\mathbb{F})$. Then

$$M_{t\times s}(\mathbb{F}) = \bigsqcup_{R\in\mathcal{E}} GL_t(\mathbb{F})\cdot R \qquad \text{and} \qquad M_{t\times s}(\mathbb{F}) = \bigsqcup_{r=0}^{\min(t,s)} GL_t(\mathbb{F}) 1_r GL_s(\mathbb{F}),$$

where

$$GL_t(\mathbb{F})R = \{PR \in M_{t\times s}(\mathbb{F}) \mid P \in GL_t(\mathbb{F})\} \qquad \text{and}$$
$$GL_t(\mathbb{F})1_r GL_s(\mathbb{F}) = \{P 1_r Q \in M_{t\times s}(\mathbb{F}) \mid P \in GL_t(\mathbb{F}), Q \in GL_s(\mathbb{F})\}.$$