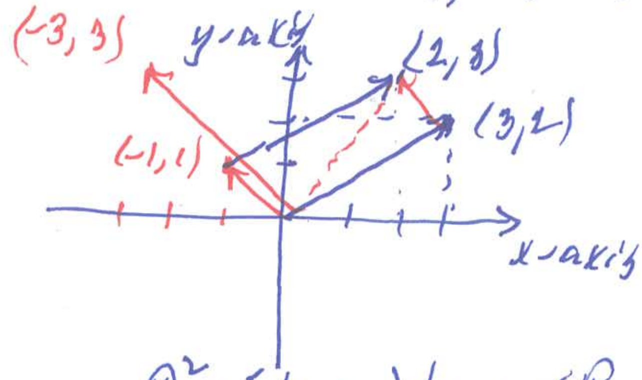
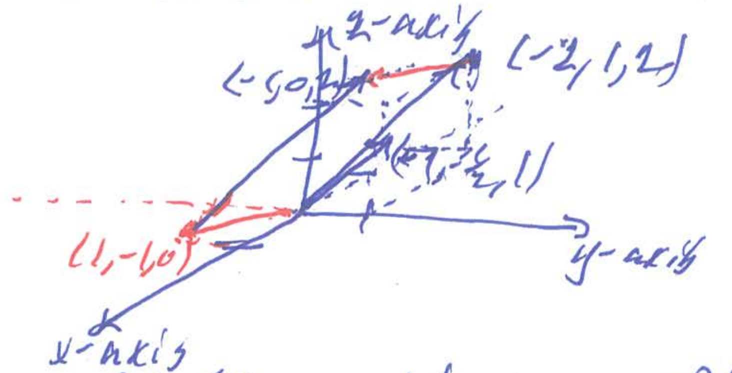
07.05.2016
Calculus head. (1)
A. Ram

# Vectors

$$\mathbb{R}^n = \{(a_1, a_2, \ldots, a_n) \mid a_1, \ldots, a_n \in \mathbb{R}\}.$$




$\mathbb{R}^2 = \{(a_1, a_2) \mid a_1, a_2 \in \mathbb{R}$

$\mathbb{R}^3 = \{(a_1, a_2, a_3) \mid a_1, a_2, a_3 \in \mathbb{R}\}.$

## Addition

$$(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n)$$
$$= (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n).$$

$(-1, 1) + (3, 2)$
$= (-1+3, 1+2) = (2, 3)$
in $\mathbb{R}^2$

and

$(1, -1, 0) + (-2, 1, 2)$
$= (1-2, -1+1, 0+2)$
$= (-1, 0, 2)$
in $\mathbb{R}^3$

## Scalar multiplication

$$c(a_1, a_2, \ldots, a_n) = (ca_1, ca_2, \ldots, ca_n)$$

$3 \cdot (-1, 1) = (3 \cdot (-1), 3 \cdot 1)$
$= (-3, 3)$
in $\mathbb{R}^2$

and

$\frac{1}{2} \cdot (-2, 1, 2) = (\frac{1}{2}(-2), \frac{1}{2} \cdot 1, \frac{1}{2} \cdot 2)$
$= (-1, \frac{1}{2}, 1)$
in $\mathbb{R}^3$

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Inner product

$$\langle (a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n) \rangle = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

$\langle (-1,1),(3,2) \rangle$
$= (-1)\cdot 3 + 1\cdot 2$
$= -3+2 = -1$
in $\mathbb{R}^2$

and

$\langle (-2,1,2),(1,-1,0) \rangle$
$= (-2)\cdot 1 + 1\cdot(-1) + 2\cdot 0$
$= -2-1+0 = -3$
in $\mathbb{R}^3$.

Length $\|(a_1, a_2, \ldots a_n)\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$
$= \sqrt{\langle (a_1, a_2, \ldots, a_n), (a_1, a_2, \ldots, a_n) \rangle}$

$\|(3,2)\| = \sqrt{3^2+2^2}$
$= \sqrt{9+4} = \sqrt{13}$.
in $\mathbb{R}^2$

and

$\|(-2,1,2)\| = \sqrt{(-2)^2+1^2+2^2}$
$= \sqrt{4+1+4} = \sqrt{9} = 3$.
in $\mathbb{R}^3$.

Distance $d((a_1, \ldots, a_n), (b_1, \ldots, b_n))$
$= \|(b_1, \ldots, b_n) - (a_1, \ldots, a_n)\|$
$= \|((b_1-a_1), (b_2-a_2), \ldots, (b_n-a_n))\|$
$= \sqrt{(b_1-a_1)^2 + (b_2-a_2)^2 + \cdots + (b_n-a_n)^2}$

In $\mathbb{R}^2$,

if $P = (3,2)$ and $Q = (-1,1)$

then $\overrightarrow{PQ} = Q - P = (-1,1) - (3,2) = (-4,-1)$

and $d(P,Q) = \|\overrightarrow{PQ}\| = \|Q-P\| = \|(-4,-1)\|$

$= \sqrt{(-4)^2 + (-1)^2} = \sqrt{16+1} = \sqrt{17}.$

In $\mathbb{R}^3$,

if $P = (-2,1,2)$ and $Q = (1,-1,0)$

then $\overrightarrow{PQ} = Q - P = (1,-1,0) - (-2,1,2) = (3,-2,-2)$

and $d(P,Q) = \|\overrightarrow{PQ}\| = \|Q-P\| = \|(3,-2,-2)\|$

$= \sqrt{3^2 + (-2)^2 + (-2)^2} = \sqrt{9+4+4} = \sqrt{17}.$

<u>Angle between vectors</u> $\theta(\vec{u},\vec{v})$

$$\cos(\theta(\vec{u},\vec{v})) = \frac{\langle \vec{u},\vec{v}\rangle}{\|\vec{u}\|\cdot\|\vec{v}\|}$$

In $\mathbb{R}^3$, if $\vec{u} = (-2,1,2)$ and $\vec{v} = (1,-1,0)$

then $$\cos(\theta(\vec{u},\vec{v})) = \frac{\langle \vec{u},\vec{v}\rangle}{\|\vec{u}\|\cdot\|\vec{v}\|} = \frac{\langle(-2,1,2),(1,-1,0)\rangle}{\sqrt{(-2)^2+1^2+2^2}\sqrt{1^2+(-1)^2+0^2}}$$

$$= \frac{-2\cdot 1 + 1\cdot(-1) + 2\cdot 0}{\sqrt{4+1+4}\sqrt{1+1+0}} = \frac{-2-1}{\sqrt{9}\sqrt{2}} = \frac{-3}{3\cdot\sqrt{2}} = \frac{-1}{\sqrt{2}}$$

$$= \frac{-\sqrt{2}}{2} = \cos\left(\frac{3\pi}{4}\right)$$

So $\theta(\vec{u},\vec{v}) = \frac{3\pi}{4}$

Unit vectors have length 1.

$\frac{1}{\|\vec{u}\|}\vec{u}$ has length 1.

In $\mathbb{R}^3$, if $\vec{u} = (-2,1,2)$ then

$$\frac{1}{\|\vec{u}\|}\cdot\vec{u} = \frac{1}{\sqrt{(-2)^2+1^2+2^2}}\cdot(-2,1,2)$$

$$= \frac{1}{\sqrt{9}}(-2,1,2) = \frac{1}{3}(-2,1,2) = \left(-\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$$

is a vector of length 1 in the same direction of $\vec{u}$.

Parallel vectors If $A = (2,0,-1)$ and $B = (1,2,-3)$ then

$$\overrightarrow{AB} = B - A = (1,2,-3) - (2,0,-1) = (-1,2,-2)$$

and $\|\overrightarrow{AB}\| = \sqrt{(-1)^2+2^2+(-2)^2} = \sqrt{1+4+4} = \sqrt{9} = 3$

and

$$\frac{1}{\|\overrightarrow{AB}\|}\overrightarrow{AB} = \frac{1}{3}(-1,2,-2) = \left(\frac{-1}{3}, \frac{2}{3}, \frac{-2}{3}\right)$$

and $\frac{-1}{\|\overrightarrow{AB}\|}\overrightarrow{AB} = \frac{-1}{3}(-1,2,-2) = \left(\frac{1}{3}, \frac{-2}{3}, \frac{2}{3}\right)$

are unit vectors parallled to $\overrightarrow{AB}$.