29.04.2026 (1)
Calculus Lect 21
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<u>Problem 4.10</u> A metal rod is heated to $110°C$ and placed into water at constant temperature $10°C$. Assume

$$\frac{dT}{dt} = -k(T-10), \quad \text{where } k \in \mathbb{R}.$$

Assume $T(0) = 110$ and $T(2) = 70$.

Find $T(t)$ and the temperature after a further 2 minutes.

<u>Solution:</u> Since $\frac{dT}{dt} = -k(T-10)$ then

$$\left(\frac{1}{T-10}\right)\frac{dT}{dt} = -k. \text{ So } \int\left(\frac{1}{T-10}\right)\frac{dT}{dt}\,dt = \int -k\,dt.$$

So

$$\log(T-10) = -kt + c, \text{ where } c \text{ is a constant.}$$

So $T - 10 = e^{-kt+c} = e^{-kt}e^{c} = Ce^{-kt}$,

where C is a constant.

So

$$T = 10 + Ce^{-kt}.$$

Since $T(0) = 110$ then

$$110 = 10 + Ce^{-k\cdot 0} = 10 + C \text{ and } C = 100.$$

Since $T(2) = 70$ then

$$70 = 10 + Ce^{-k\cdot 2} = 10 + 100e^{-k\cdot 2}$$

and $60 = 100e^{-2k}$. So $\frac{6}{10} = e^{-2k}$.

So $\log\left(\frac{3}{5}\right) = -2k$ and $k = -\frac{1}{2}\log\left(\frac{3}{5}\right)$.

So

$$T = 10 + 100e^{-\frac{1}{2}\log\left(\frac{5}{3}\right)t}$$

So

$$T(4) = 10 + 100e^{-\frac{1}{2}\log\left(\frac{5}{3}\right)\cdot 4}$$

$$= 10 + 100\left(\frac{3}{5}\right)^2 = 10 + 100\cdot\frac{9}{25} = 10 + 4\cdot 9$$

$$= 10 + 36 = 46^\circ C.$$

<u>Problem (4.2)(2)</u> Solve $\frac{d^2y}{dx^2} = \frac{1}{(x+1)^2}$.

Since $\frac{d\left(\frac{dy}{dx}\right)}{dx} = (x+1)^{-2}$ then

$$\frac{dy}{dx} = (-1)(x+1)^{-1} + c_1$$

$$= \frac{-1}{x+1} + c_1,$$ where c_1 is a constant.

Since $\frac{dy}{dx} = \frac{-1}{x+1} + c_1$ then $y = -\log(x+1) + c_1x + c_2$,

where c_1 and c_2 are constants.

Problem 4.1(4) Verify that

$y = e^{-2x} + e^{3x}$ is a solution of $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0.$

Since $y = e^{-2x} + e^{3x}$

then $\frac{dy}{dx} = -2e^{-2x} + 3e^{3x}$

Then

and $\frac{d^2y}{dx^2} = 4e^{-2x} + 9e^{3x}.$

$$\begin{aligned}\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y &= 4e^{-2x} + 9e^{3x} \\ &\quad + 2e^{-2x} - 3e^{3x} \\ &\quad - 6e^{-2x} - 6e^{3x} \\ &= 0e^{-2x} + 0e^{3x} = 0. \ //.\end{aligned}$$

Problem 4.3(7) Solve $\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{x}.$

Since $\frac{1}{\sqrt{1-y^2}} \frac{dy}{dx} = \frac{1}{x}$ then

$$\int \frac{1}{\sqrt{1-y^2}} \frac{dy}{dx}\, dx = \int \frac{1}{x}\, dx.$$

So

$\arcsin(y) = \log|x| + c$, where c is a constant.

So $y = \sin(\log(x) + c)$

$= \sin(\log(x))\cos(c) + \cos(\log(x))\sin(c).$

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Problem 4.4(3) Find y if

$$\frac{dy}{dx} = 2y-6 \quad \text{and} \quad y(0)=1.$$

Since $\frac{1}{(2y-6)}\frac{dy}{dx} = 1$ then $\int \frac{1}{(2y-6)}\frac{dy}{dx}dx = \int 1\,dx$.

So $\frac{1}{2}\log(2y-6) = x+c$ and $\log(2y-6) = 2x+2c$.

So $2y-6 = e^{2x+2c} = e^{2c}e^{2x} = Ce^{2x}$,

where c and C are constants.

So $2y = 6+Ce^{2x}$ and $y = 3+\frac{1}{2}Ce^{2x} = 3+Ke^{2x}$.

where K is a constant.

If $x=0$ then $y=1$ and $1 = 3+Ke^{2\cdot 0} = 3+K$.

So $K=-2$ and $y = 3-2e^{2x}$.