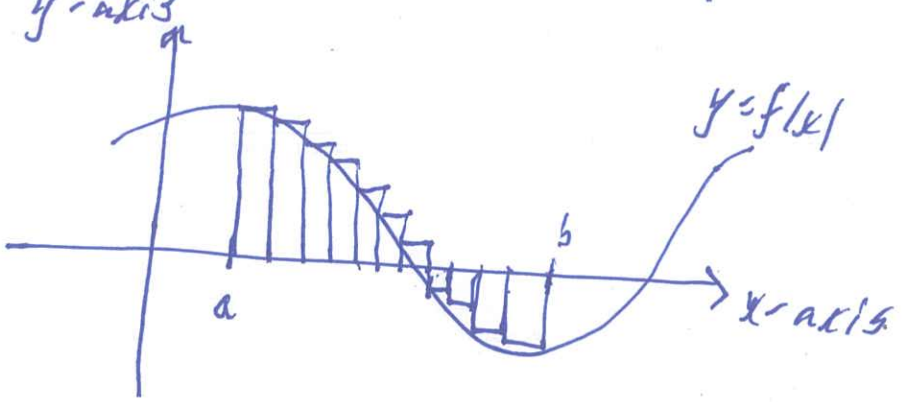
04.05.2026
Calculus Lect. 24 (1)
A. Ram

# Definite integral

$$\int_a^b f(x)\,dx = \lim_{N\to\infty}\left(f(a)\frac{1}{N} + f\left(a+\frac{1}{N}\right)\frac{1}{N} + \cdots + f\left(b-\frac{1}{N}\right)\frac{1}{N}\right)$$




So

$$\int_a^b f(x)\,dx = \lim_{N\to\infty}\left(\begin{array}{l}\text{add up the areas of the} \\ \text{little boxes of width } \frac{1}{N} \\ \text{and height } f\left(x+\frac{k}{N}\right)\end{array}\right)$$

## Fundamental Theorem of Calculus

Let $A(x) = \int f(x)\,dx$ (i.e. $\frac{dA}{dx} = f$).

Then

$$\int_a^b f(x)\,dx = A(b) - A(a).$$

Write $A(x)\Big]_{x=a}^{x=b} = A(b) - A(a)$ so that

$$\int_a^b f(x)\,dx = A(x)\Big]_{x=a}^{x=b}.$$

<u>Problem</u> Find $\int_0^1 x^2\,dx$.

Let $A(x) = \int x^2\,dx = \frac{1}{3}x^3 + c$, where $c$ is a constant.
Using the fundamental theorem of calculus,

$$\int_0^1 x^2\,dx = A(x)\Big]_{x=0}^{x=1} = \frac{1}{3}x^3 + c\Big]_{x=0}^{x=1} = \left(\frac{1}{3}1^3 + c\right) - \left(\frac{1}{3}0^3 + c\right)$$

$$= \frac{1}{3}$$

<u>Problem</u> Find $\int_1^2 (2e^x - 21x^{-8})\,dx$.

Let $A(x) = \int (2e^x - 21x^{-8})\,dx = 2e^x + 3x^{-7} + c$, where $c$ is a constant. Using the fundamental theorem of calculus

$$\int_1^2 (2e^x - 21x^{-8})\,dx = A(x)\Big]_{x=1}^{x=2} = \left(2e^x + 3x^{-7} + c\right)\Big]_{x=1}^{x=2}$$

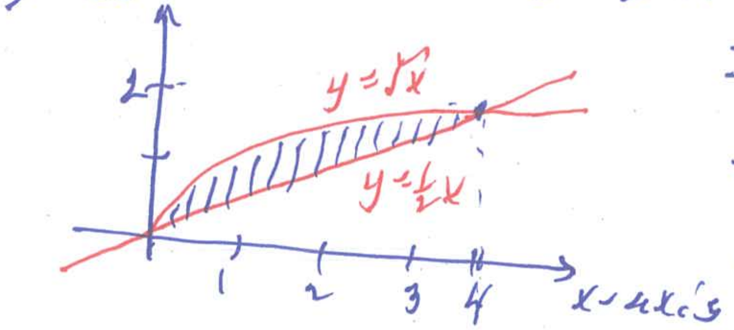
$$= \left(2e^2 + 3\cdot 2^{-7} + c\right) - \left(2e^1 + 3\cdot 1^{-7} + c\right)$$

$$= 2e^2 - 2e + 3\left(\frac{1}{2^7} - 1\right) = 2e(e-1) + 3\cdot\left(\frac{1}{128} - 1\right)$$

$$= 2e(e-1) + 3\cdot\left(-\frac{127}{128}\right) = 2e(e-1) - \frac{381}{128}.$$

**Problem** Find the area between

$y=\sqrt{x}$ and $y=\frac{1}{2}x$.




If $y^2=x$ and $y=\frac{1}{2}x$ then $\left(\frac{1}{2}x\right)^2=x$ and $0=\frac{1}{4}x^2-x=\left(\frac{1}{4}x-1\right)x$ so that $x=0$ or $x=4$.

Add up slices $y_{top}-y_{bottom}$ from $x=0$ to $x=4$

$dx$

$$\int_{x=0}^{x=4}(y_{top}-y_{bottom})\,dx=\int_{x=0}^{x=4}\left(\sqrt{x}-\frac{1}{2}x\right)dx=\int_{x=0}^{x=4}\left(x^{\frac{1}{2}}-\frac{1}{2}x\right)dx$$

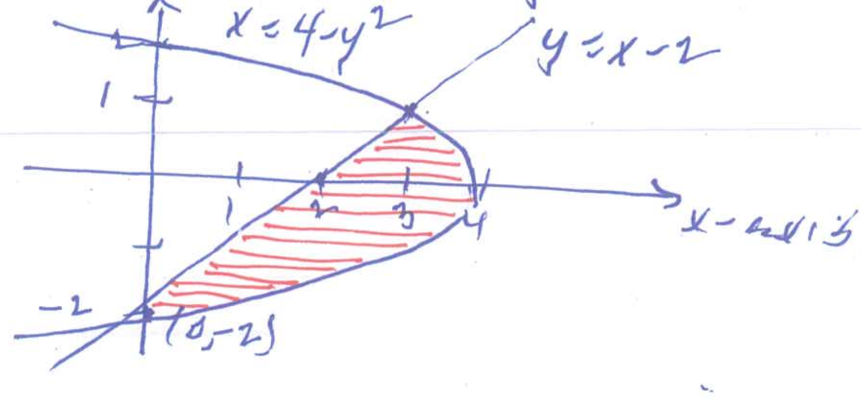
$$=\int_{x=0}^{x=4}\left(\frac{2}{3}\cdot\frac{3}{2}x^{\frac{1}{2}}-\frac{1}{2}\cdot\frac{1}{2}\cdot 2x\right)dx=\frac{2}{3}x^{3/2}-\frac{1}{4}x\Big]_{x=0}^{x=4}$$

$$=\left(\frac{2}{3}4^{3/2}-\frac{1}{4}\cdot 4\right)-\left(\frac{2}{3}0^{3/2}-\frac{1}{4}\cdot 0\right)=\frac{2}{3}\cdot 2^3-4$$

$$=\frac{16}{3}-\frac{12}{3}=\frac{4}{3}.$$

**Problem** Find the area enclosed by

$y=x-2$ and $x=4-y^2$



If $x = 4-y^2$ and $y = x-2$ then

$$y = 4-y^2-2 = 2-y^2. \text{ So } y^2+y-2=0.$$

So $(y+2)(y-1)=0$

Add slices $\overbrace{\qquad\qquad}^{x_{right}-x_{left}} dy$ from $y=-2$ to $y=1$.

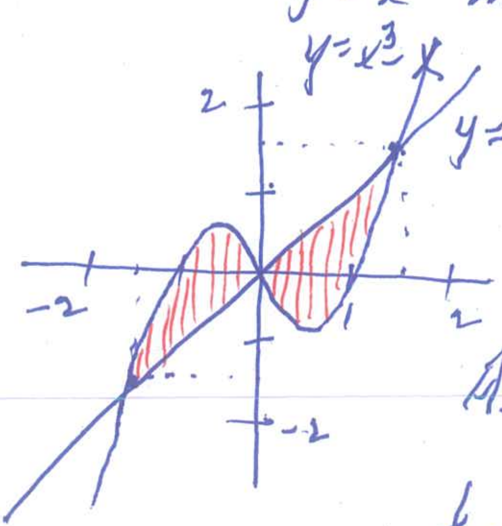
$$\int_{y=-2}^{y=1} (x_{right}-x_{left})\,dy = \int_{y=-2}^{y=1} ((4-y^2)-(y+2))\,dy$$

$$= \int_{y=-2}^{y=1} (-y^2+4-y-2)\,dy = \int_{y=-2}^{y=1} (-y^2-y+2)\,dy$$

$$= -\frac{1}{3}y^3 - \frac{1}{2}y^2 + 2y \Big]_{y=-2}^{y=1} = \left( \left(-\frac{1}{3}\cdot 1^3 - \frac{1}{2}\cdot 1^2 + 2\cdot 1\right) - \left(-\frac{1}{3}(-2)^3 - \frac{1}{2}(-2)^2 + 2\cdot(-2)\right)\right)$$

$$= -\frac{1}{3} - \frac{1}{2} + 2 - \frac{8}{3} + 2 + 4 = 8 - 3 - \frac{1}{2} = 5 - \frac{1}{2} = \frac{9}{2}.$$

<u>Problem</u> Find the area enclosed by
$y = x$ and $y = x^3 - x$.




If $y = x$ and $y = x^3 - x = x(x^2-1)$
then $x = x(x^2-1)$ or $x^3 - 2x = 0$
So $x = \sqrt{2}$ or $x = -\sqrt{2}$ or $x = 0$.

Add up slices $\Big\}\, y_{top} - y_{bottom}$
$dx$

from $x = 0$ ~~th~~ $x = \sqrt{2}$ and multiply by 2

$$\text{Area} = 2\cdot\int_{x=0}^{x=\sqrt{2}} (y_{top} - y_{bottom})\,dx$$

$$= 2\int_{x=0}^{x=\sqrt{2}} (x-(x^3-x))\,dx = 2\int_{x=0}^{x=\sqrt{2}} (-x^3+2x)\,dx$$

$$= 2\left(-\frac{1}{4}x^4 + x^2\right)\Big|_{x=0}^{x=\sqrt{2}} = 2\left(-\frac{1}{4}(\sqrt{2})^4+(\sqrt{2})^2\right) - 2\left(-\frac{1}{4}\cdot 0^4+0^2\right)$$

$$= 2\left(\left(\frac{-1}{4}\right)\cdot 4 + 2 - 0 - 0\right) = 2\cdot(-1+2) = 2\cdot 1 = 2.$$