Q4 Let A and B be subsets of $\mathbb{R}$

Let $f: A \to B$ be a function.

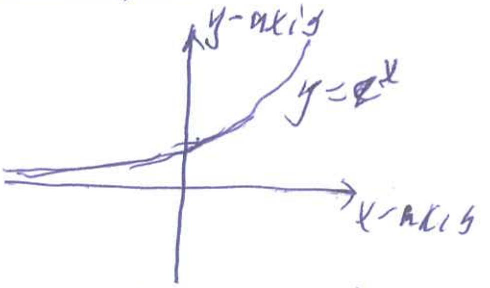
The function f is monotone increasing if f satisfies

if $a_1, a_2 \in A$ and $a_1 < a_2$ then $f(a_1) < f(a_2)$.

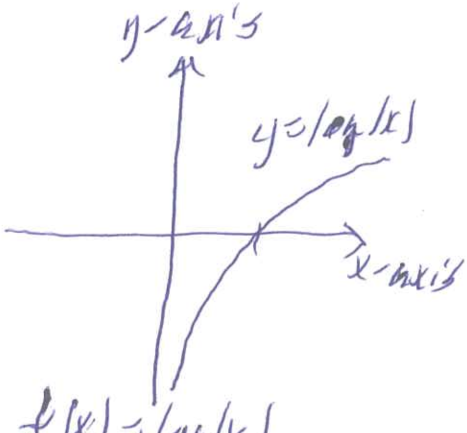
The function f is injective if f satisfies

if $a_1, a_2 \in A$ and $a_1 \neq a_2$ then $f(a_1) \neq f(a_2)$

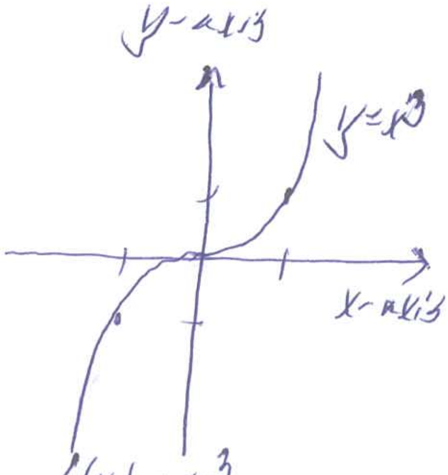
Examples



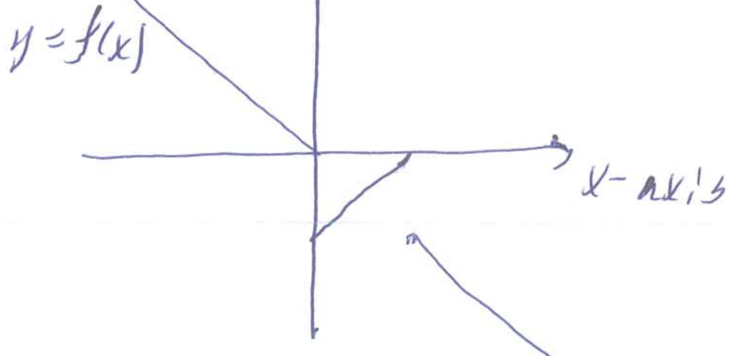
$y = e^x$ is monotone and injective



$f(x) = \log(x)$ is monotone and injective



$f(x) = x^3$ is monotone and injective



$f(x)$ is injective and not monotone

Let $k, t \in \mathbb{R}_{>1}$.

Q4 (b) To show: If f and g are monotone increasing then $kf + tg$ is monotone increasing.

Assume f and g are monotone increasing.

To show: $kf+tg$ is monotone increasing.

To show: If $a_1, a_2 \in \mathbb{R}$ and $a_1 < a_2$ then

$$(kf+tg)(a_1) < (kf+tg)(a_2)$$

Assume $a_1, a_2 \in \mathbb{R}$ and $a_1 < a_2$.

To show: $(kf+tg)(a_1) < (kf+tg)(a_2)$

$$
\begin{aligned}
(kf+tg)(a_1) &= kf(a_1) + tg(a_1) \\
&< kf(a_2) + tg(a_1), && \text{since } k>1 \text{ and } f(a_1) < f(a_2) \\
&< kf(a_2) + tg(a_2), && \text{since } t>1 \text{ and } g(a_1) < g(a_2) \\
&= (kf+tg)(a_2).
\end{aligned}
$$

So $(kf+tg)(a_1) < (kf+tg)(a_2)$

So $kf+tg$ is monotone increasing.

Q4(a) To show: h is not monotone increasing.

x	1	2	3	4	5
$h(x)$	2	3	1	4	5

Let $a_1 = 2$ and $a_2 = 3$.

Then $a_1 < a_2$ and $h(a_1) = 3 > 1 = h(a_2)$

So h is not monotone increasing

Q3 Let $P(z) = z^3 + a_1 z^2 + a_2 z + a_3$ with

$a_1, a_2, a_3 \in \mathbb{R}$.

If $\alpha \in \mathbb{C}$ is a root of $P(z)$ then $\bar{\alpha}$ is a root of $P(z)$ also.

Since 0 and $1+i$ are roots of $P(z)$ then $1-i$ is also a root of $P(z)$ and

$$P(z) = (z-0)(z-(1+i))(z-(1-i))$$
$$= z\left(z^2 - (1-i)z - (1+i)z + (1+i)(1-i)\right)$$
$$= z\left(z^2 - 2z + (1^2 + (+1)^2)\right)$$
$$= z(z^2 - 2z + 2)$$

Irreducible factors in $\mathbb{R}[z]$ are z and $z^2 - 2z + 2$.

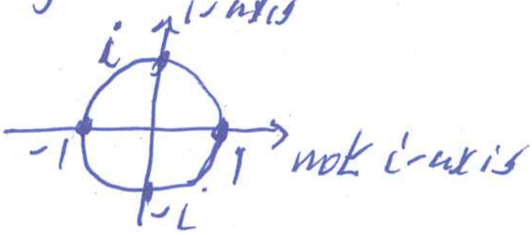
Q1 $A = \{ z \in \mathbb{C} \mid z^4 = -16 \}$

$= \{ z \in \mathbb{C} \mid z^4 = 2^4 e^{i\pi} \}$

$= \{ z \in \mathbb{C} \mid \frac{z^4}{2^4 e^{i\pi}} = 1 \} = \{ z \in \mathbb{C} \mid \left(\frac{z}{2e^{i\pi/4}}\right)^4 = 1 \}.$

The 4th roots of unity are

$1, i, -1, -i$



So

$$A = \left\{ z \in \mathbb{C} \;\middle|\; \frac{z}{2e^{i\pi/4}} = 1 \text{ or } \frac{z}{2e^{i\pi/4}} = i \text{ or } \frac{z}{2e^{i\pi/4}} = -1 \text{ or } \frac{z}{2e^{i\pi/4}} = -i \right\}$$

$$= \{ z = 2e^{i\pi/4} \text{ or } z = i2e^{i\pi/4} \text{ or } z = -2e^{i\pi/4} \text{ or } z = -i2e^{i\pi/4} \}$$

$$= \{ 2(\tfrac{1}{2} + \tfrac{1}{2}i),\ i2(\tfrac{1}{2} + \tfrac{1}{2}i),\ -2(\tfrac{1}{2} + \tfrac{1}{2}i),\ -i2(\tfrac{1}{2} + \tfrac{1}{2}i) \}$$

$$= \{ 1+i,\ -1+i,\ -1-i,\ 1-i \}$$

Q1 (ii) Let A be the set of natural numbers that have remainder 2 when divided by 5

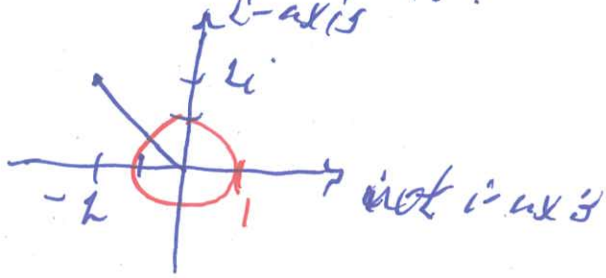
$$A = \{ x \in \mathbb{N} \mid \text{there exists } k \in \mathbb{Z} \text{ such that } x = 2k+5 \}$$

Q1 (iii)

$$\{ z \in \mathbb{C} \mid |z - w| = 1 \}$$
$$= \{ z \in \mathbb{C} \mid |z - (3+4i)| = 1 \}$$
$$= \{ z \in \mathbb{C} \mid z \text{ is distance } 1 \text{ from } 3+4i \}$$
= circle of radius 1 centred at $3+4i$.

Q1 (iv)

$$z = -2+2i$$
$$= 4\left(\tfrac{-1}{2} + \tfrac{1}{2} i\right)$$
$$= 4e^{i3\pi/4}$$



Q1 (v)

$$\frac{d\, e^y}{dx} = e^y \frac{dy}{dx}$$

Q1 (vi)

$$\frac{z}{z+1} = \frac{-2+2i}{-2+2i+1} = \frac{(-2+2i)}{(-1+2i)} \frac{(-1-2i)}{(-1-2i)}$$
$$= \frac{2+4i-2i-4i^2}{1+4} = \frac{6+2i}{5} = \frac{6}{5} + \frac{2}{5} i.$$

Q1 (vii)

$$z^{100} = (-2+2i)^{100} = \left(4e^{i3\pi/4}\right)^{100} = 4^{100} e^{i\frac{3\cdot 100}{4}\pi}$$
$$= 4e^{i3\cdot 25\pi} = 4e^{i75\pi} = 4e^{i\pi}. \text{ So } \operatorname{Arg}(z^{100}) = \pi.$$