<u>Problem 1.38 (6)</u> Find the implied domain and range of the expression

$$r(x) = \log(\sqrt{5-x} + \sqrt{x+1})$$

Let $S = \{x \in \mathbb{R} \mid x \le 5 \text{ and } x \ge -1 \text{ and } \sqrt{5-x} + \sqrt{x+1} > 0\}$

$= \{x \in \mathbb{R} \mid x \in [-1,5]\} = [-1,5]$,

since $\sqrt{5-5} + \sqrt{5+1} = \sqrt{6} > 0$

and $\sqrt{5-(-1)} + \sqrt{(-1)+1} = \sqrt{6} > 0$.

So
$$\begin{aligned} r: S &\to \mathbb{R} \\ x &\mapsto \log(\sqrt{5-x} + \sqrt{x+1}) \end{aligned}$$
is a function

The range of $r$ is

$r(S) = \{r(x) \mid x \in [-1,5]\}$

$= \{\log(\sqrt{5-x} + \sqrt{x+1}) \mid x \in [-1,5]\}$

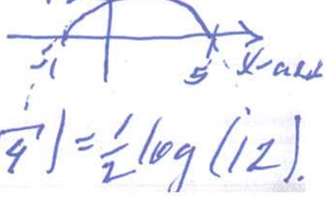
$= \{\frac{1}{2}\log((\sqrt{5-x} + \sqrt{x+1})^2) \mid x \in [-1,5]\}$

$= \{\frac{1}{2}\log(5-x + 2\sqrt{5-x}\sqrt{x+1} + x+1) \mid x \in [-1,5]\}$

$= \{\frac{1}{2}\log(6 + 2\sqrt{5+4x-x^2}) \mid x \in [-1,5]\}$

Then $5+4x-x^2$ has zeros at $x=-1$ and $x=5$

and a minimum at $x=2$




Since $\log(a) \to -\infty$ as $a \to 0$ and

and $\frac{1}{2}\log(6+2\sqrt{5+4\cdot 2-2^2}) = \frac{1}{2}\log(6+2\sqrt{9}) = \frac{1}{2}\log(12)$.

then

$$r(S) = [-\infty, \tfrac{1}{2}\log(12)]$$
$$= [-\infty, \log(12^{\frac{1}{2}})] = [-\infty, \log(\sqrt{12})]$$
$$= [-\infty, \log(2\sqrt{3})]$$

<u>Problem 1.38(5)</u> Find the implied domain and range of the expression

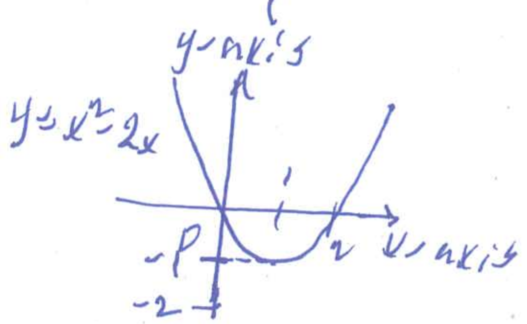
$$q(x) = \sqrt{\frac{1}{x} - \frac{1}{x-2}} = \sqrt{\frac{x-2-x}{x(x-2)}} = \sqrt{\frac{-2}{x(x-2)}}$$

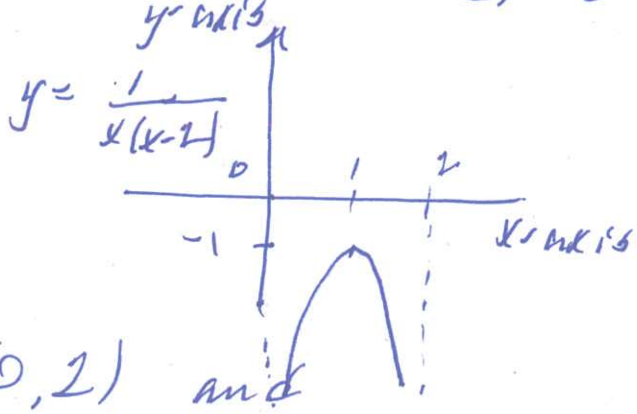
Let

$$S = \left\{x \in \mathbb{R} \mid x \neq 0 \text{ and } x \neq 2 \text{ and } \frac{-2}{x(x-2)} \geq 0\right\}$$

$$= \left\{x \in \mathbb{R} \mid x \neq 0 \text{ and } x \neq 2 \text{ and } -2 \geq x(x-2) \text{ and } x(x-2) > 0\right\}$$

$$\cup \left\{x \in \mathbb{R} \mid x \neq 0 \text{ and } x \neq 2 \text{ and } -2 \leq x(x-2) \text{ and } x(x-2) < 0\right\}$$




So $S = \emptyset \cup (0,2) = (0,2)$ and

$$\begin{aligned} q: S &\to \mathbb{R} \\ x &\mapsto \sqrt{\frac{1}{x} - \frac{1}{x-2}} \end{aligned}$$ is a function

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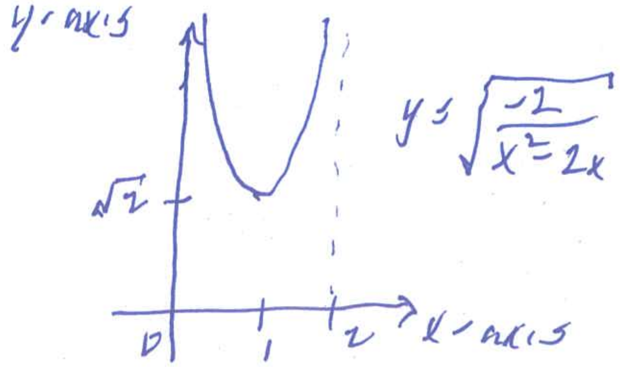


So the range is $g(S) = [\sqrt{2}, \infty)$.

---

Three types of proof:

<u>Type I</u>: LHS = RHS.

<u>Proof</u>: LHS = ... = ... = ... = ... = STUFF

RHS = ... = ... = the same STUFF.

<u>Type II</u> To show: If ___A___ then ___B___

Assume ___A___

To show: ___B___

<u>Type III</u> To show: there exists ___C___ such that ___D___.

Let C = ....

To show: ___D___

<u>Synonyms</u>: if = for all = for each = $\forall$

then = so = hence = therefore = $\therefore$

there exists = for some = $\exists$

<u>IMPORTANT</u>: Translate Math to English and English to Math.

Cartoons are equally important.

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## How to study for exams

(1) Make up exams..

(2) Do exams.

(3) Mark exams

Main goal:
Learn how to optimize your marks

Very important: Time management

Problem 1.38 (A4) Find the implied domain and range of the expression

$$p(x) = \frac{\sqrt{x^2-9}}{x-4}.$$

Let
$$S = \{x\in\mathbb{R} \mid x\ne 4 \text{ and } x^2-9\ge 0\}$$
$$= \{x\in\mathbb{R} \mid x\ne 4 \text{ and } x^2\ge 9\}$$
$$= \{x\in\mathbb{R} \mid x\ne 4 \text{ and } x\le -3\} \cup \{x\in\mathbb{R} \mid x\ne 4 \text{ and } x\ge 3\}$$
$$= (-\infty,-3] \cup [3,4) \cup (4,\infty).$$

So
$$\begin{aligned} p: S &\to \mathbb{R} \\ x &\mapsto \frac{\sqrt{x^2-9}}{x-4} \end{aligned}$$
is a function.

The range of $p$ is $p(S)$.

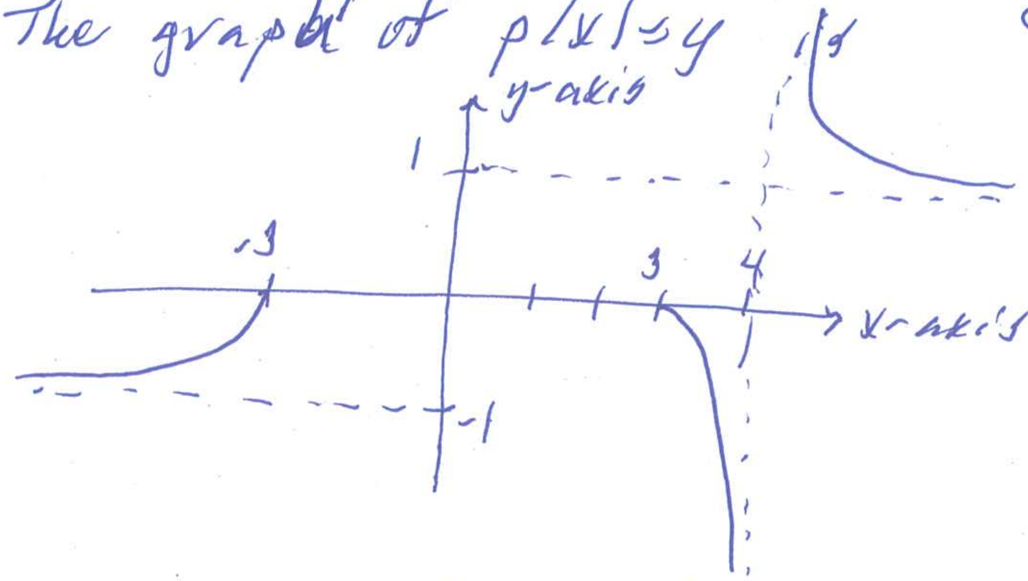
$$\frac{\sqrt{x^2-9}}{x-4} = \begin{cases} \sqrt{\dfrac{x^2-9}{(x-4)^2}}, & \text{if } x\ge 4 \\ -\sqrt{\dfrac{x^2-9}{(x-4)^2}}, & \text{if } x<4 \end{cases}$$

and
$$\sqrt{\frac{x^2-9}{(x-4)^2}} = \sqrt{\frac{x^2-9}{x^2-8x+16}} = \sqrt{\frac{1-\frac{9}{x^2}}{1-\frac{8}{x}+\frac{16}{x^2}}}$$

As $x\to\infty$ then $\sqrt{\dfrac{1-9/x^2}{1-\frac{8}{x}+\frac{16}{x^2}}} \to \sqrt{\dfrac{1-0}{1-0+0}} = 1.$

As $x\to-\infty$ then $-\sqrt{\dfrac{1-9/x^2}{1-8/x+16/x^2}} \to -\sqrt{\dfrac{1-0}{1-0+0}} = -1.$

The graph of $p(x)=y$




So $p(S) = p((-\infty,-3]) \cup p((3,4)) \cup p((4,\infty))$

$= (-1,0] \cup (0,-\infty) \cup (\infty,1)$

$= (-\infty,0] \cup (1,\infty)$.

<u>Problem 1.38 (3)</u> Find the implied domain and range of

$$h(x) = \log(\sqrt{2x+5}-1)$$

Let $S = \{x \in \mathbb{R} \mid 2x+5 \geq 0 \text{ and } \sqrt{2x+5}-1 > 0\}$

$= \{x \in \mathbb{R} \mid x \geq -\frac{5}{2} \text{ and } \sqrt{2x+5} > 1\}$

$= \{x \in \mathbb{R} \mid x \geq -\frac{5}{2} \text{ and } 2x+5 > 1^2\}$

$= \{x \in \mathbb{R} \mid x \geq -\frac{5}{2} \text{ and } x > -\frac{4}{2}\}$

$= (-2,\infty)$.

So $h: (-2,\infty) \to \mathbb{R}$ is a function.
$x \mapsto \log(\sqrt{2x+5}-1)$

$h(-2) = \log(\sqrt{2(-2)+5}-1) = \log(1-1) = \log(0)$

As $x \to -2$ then $\log(\sqrt{2x+5}-1) \to -\infty$.

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As $x \to \infty$ then $\log(\sqrt{2x+6}-1) \to \infty$.

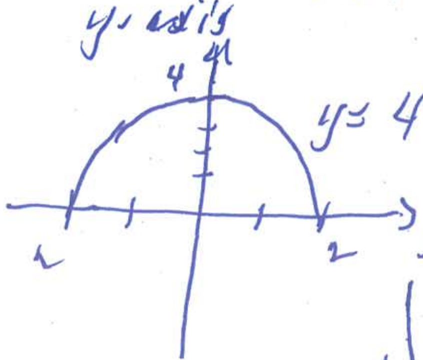
So $h(S) = (-\infty, \infty) = \mathbb{R}$ is the range of $h$.

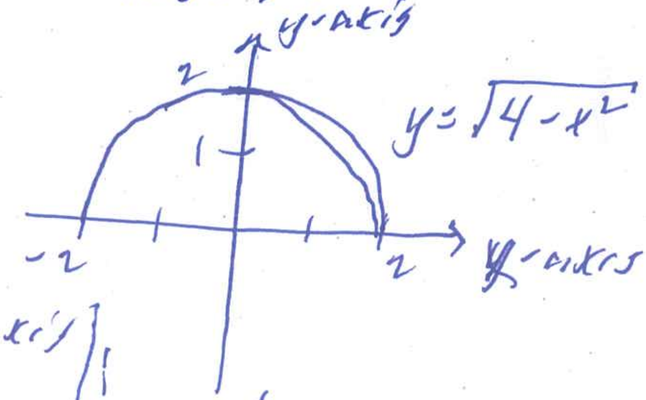
<u>Problem 1.38 (2)</u> Find the implied domain and range of the expression

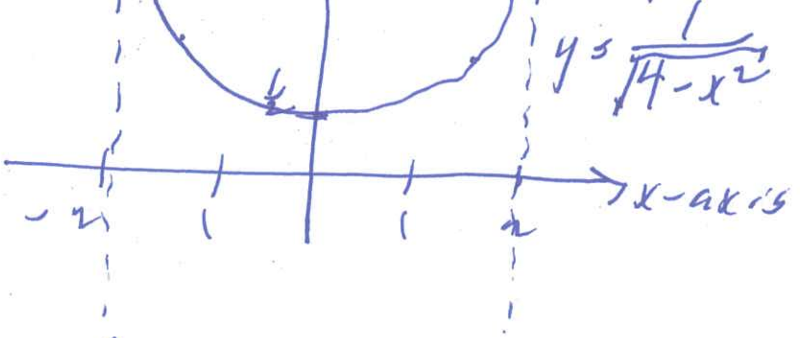
$$g(x) = \frac{1}{\sqrt{4-x^2}}$$

Let $S = \{x \in \mathbb{R} \mid 4-x^2 \geq 0 \text{ and } \sqrt{4-x^2} \neq 0\}$

$= \{x \in \mathbb{R} \mid x^2 < 4\} = (-2, 2)$.




So $g: S \to \mathbb{R}$, $x \mapsto \frac{1}{\sqrt{4-x^2}}$ is a function

with range $g(S) = g((-2,2)) = [\frac{1}{2}, \infty)$.

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<u>Problem 1.38(1)</u> Find the implied domain and range of the expression

$$f(x) = \sqrt{\frac{x-3}{x+1}}$$

Let $S = \{x \in \mathbb{R} \mid x \neq -1 \text{ and } \frac{x-3}{x+1} \geq 0\}$

$= \{x \in \mathbb{R} \mid x \neq -1 \text{ and } x-3 \geq 0 \text{ and } x+1 > 0\}$
$\cup \{x \in \mathbb{R} \mid x \neq -1 \text{ and } x-3 \leq 0 \text{ and } x+1 < 0\}$

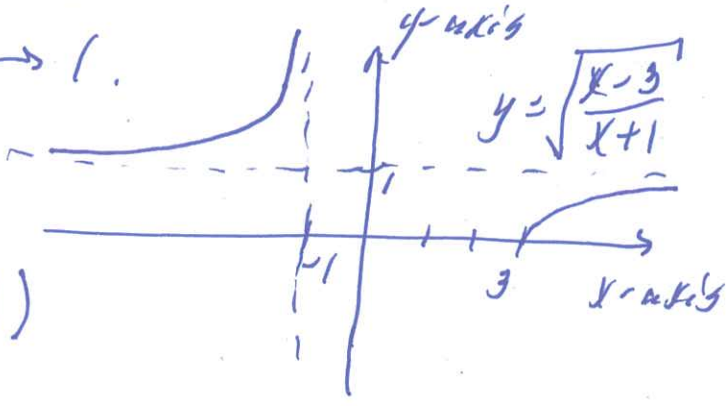
$= \{x \in \mathbb{R} \mid x \neq -1 \text{ and } x \geq 3 \text{ and } x > -1\}$
$\cup \{x \in \mathbb{R} \mid x \neq -1 \text{ and } x \leq 3 \text{ and } x < -1\}$

$= [3, \infty) \cup (-\infty, -1) = (-\infty, -1) \cup [3, \infty).$

So $\begin{matrix} f : S \longrightarrow \mathbb{R} \\ x \longmapsto \sqrt{\frac{x-3}{x+1}} \end{matrix}$ is a function.

As $x \to \infty$ then $\sqrt{\frac{x-3}{x+1}} = \sqrt{\frac{1-\frac{3}{x}}{1+\frac{1}{x}}} \longrightarrow \sqrt{\frac{1-0}{1+0}} = 1$

As $x \to -\infty$ then $\sqrt{\frac{x-3}{x+1}} \to 1.$




The range is

$f(S) = f((-\infty, -1)) \cup f([3, \infty))$

$= (1, \infty) \cup [0, 1)$

$= [0, 1) \cup (1, \infty).$