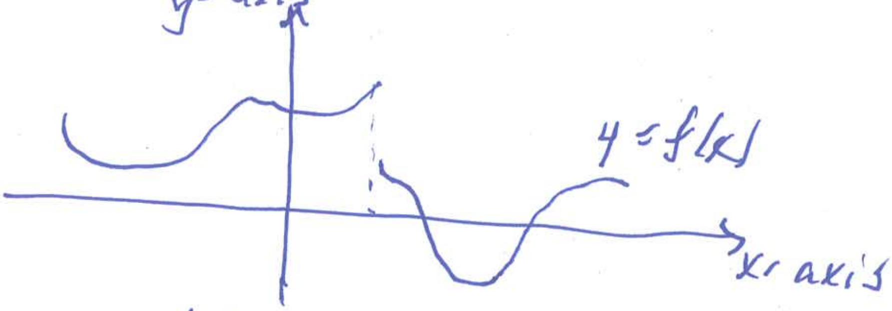
A function is <u>continuous at $x=a$</u> if it doesn't jump at $x=a$

i.e. if $\lim_{x\to a} f(x) = f(a)$

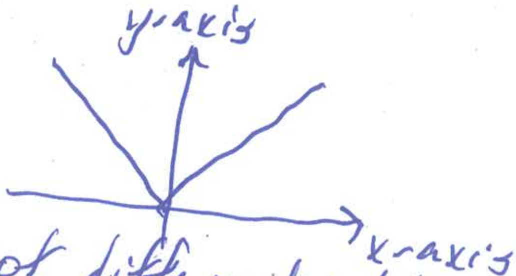


Not continuous at $x=a$.

A function is <u>differentiable at $x=a$</u> if

$$f'(a) = \lim_{x\to a} \frac{f(a+h)-f(a)}{h} \quad \text{exists}$$

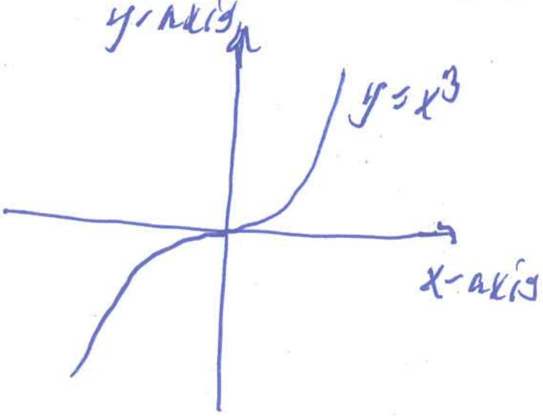
<u>Example</u> $f(x) = |x|$

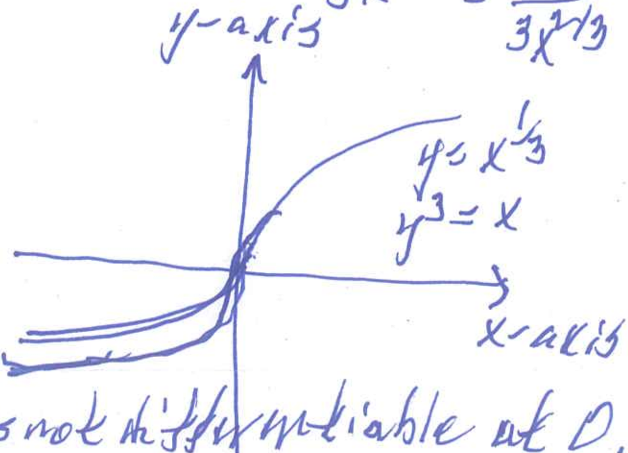


$f'(0)$ does not exist; <u>not</u> differentiable at $x=0$.

<u>Example</u> $f(x) = x^{1/3}$.

$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$



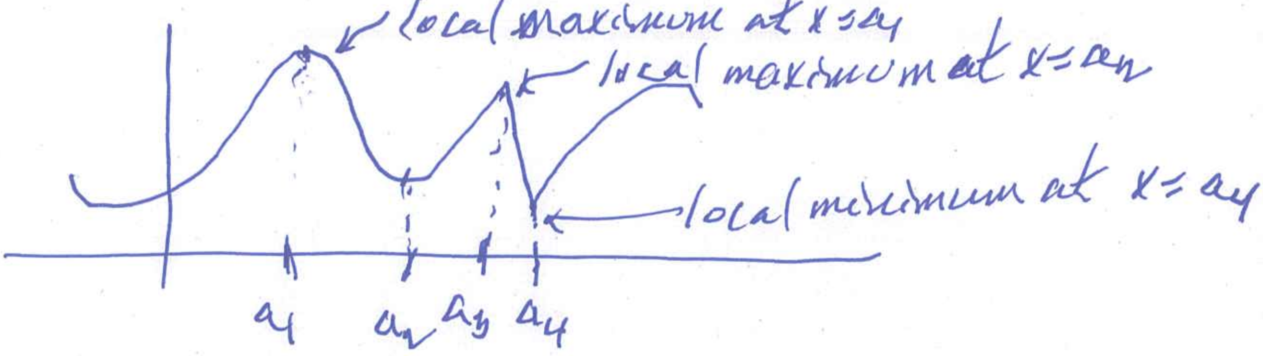


f is not differentiable at 0.

$f'(0)$ does not exist.
(it is infinite).

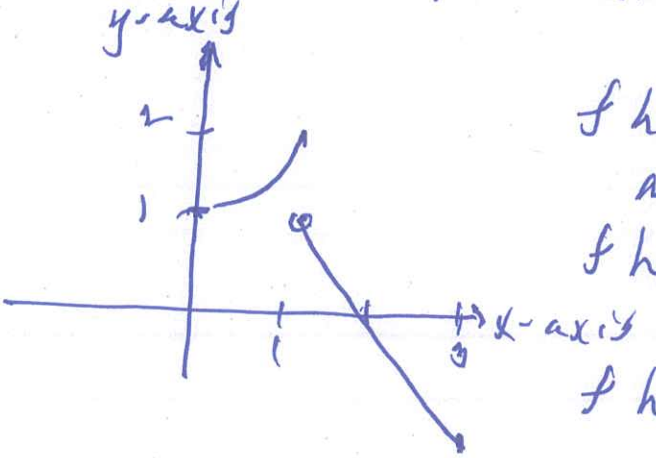
A <u>local maximum</u> is a point $x=a$ where $f(a)$ is bigger than the $f(x)$ around it.

A <u>local minimum</u> is a point $x=a$ where $f(a)$ is smaller than the $f(x)$ around it.



<u>Example</u> $f: [0,3] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x^2+1, & \text{if } 0 \le x \le 1, \\ 2-x, & \text{if } x > 2. \end{cases}$$



f has a local minimum at $x=0$

f has a local maximum at $x=1$

f has a local minimum at $x=3$.

f is <u>not</u> differentiable at $x=1$.

Example: Find horizontal and vertical tangent lines for the graph of $y^2 = x^3+1$.

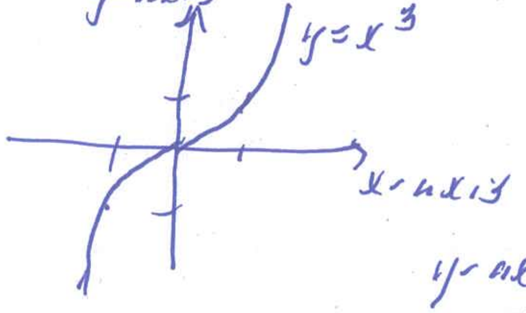
Take the derivative of both sides:

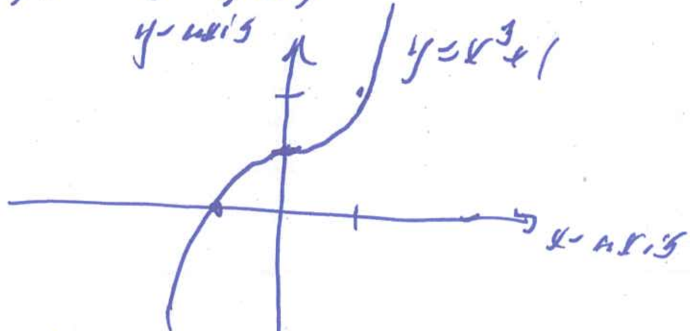
$$2y\frac{dy}{dx} = 3x^2. \quad \text{So} \quad \frac{dy}{dx} = \frac{3x^2}{2y} = \frac{3x^2}{2\sqrt{x^3+1}}.$$

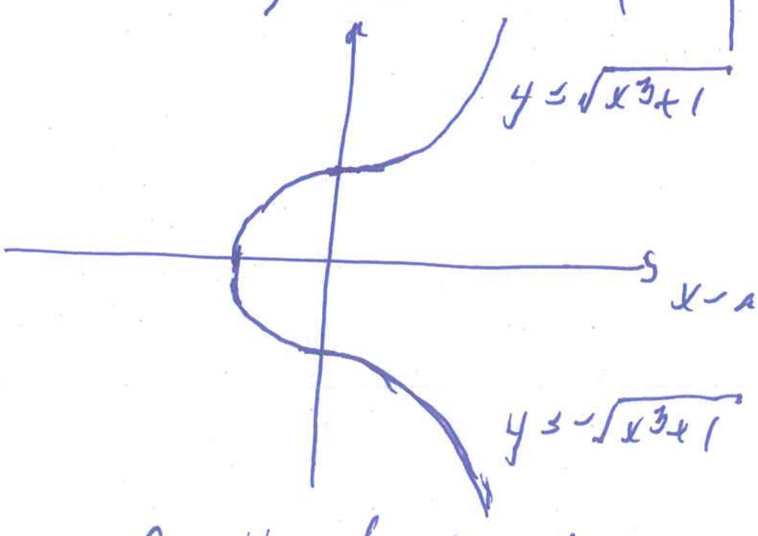
If $x=0$ then $\frac{dy}{dx}=0$ and $y^2 = 0^3+1$ so that $y=1$ or $y=-1$.

There is probably a horizontal (slope 0) tangent line at $(x,y)=(0,1)$ and $(x,y)=(0,-1)$.

Since $2\sqrt{x^3+1}$ is never 0 ~~then~~ unless $x=-1$ then probably the only vertical (slope ∞) tangent line is at $(x,y)=(-1,0)$







Graph of $y^2 = x^3+1$.

Example: Find horizontal and vertical tangent lines of $x^2+xy+y^2=9$.

Take the derivative of both sides:

$$2x+x\frac{dy}{dx}+y+2y\frac{dy}{dx}=0.$$

So $(x+2y)\frac{dy}{dx}=-y-2x$

So $$\frac{dy}{dx}=\frac{-y-2x}{x+2y}$$

Case 1: If $-y-2x$ is 0 then $\frac{dy}{dx}=0$ and we should get a slope 0 tangent line.

If $-y-2x=0$ and $x^2+xy+y^2=9$

then $y=-2x$ and $x^2+x(-2x)+(-2x)^2=9$.

So $x^2-2x^2+4x^2=9$. So $3x^2=9$ and $x^2=3$.

So $x=\sqrt{3}$ or $x=-\sqrt{3}$.

If $x=\sqrt{3}$ then $y=-2\sqrt{3}$

If $x=-\sqrt{3}$ then $y=-2\cdot(-\sqrt{3})=2\sqrt{3}$

Case 2: If $x+2y$ is 0 then $\frac{dy}{dx}$ is probably infinite and we should get a slope ∞ (vertical) tangent line.

If $x+2y=0$ and $x^2+xy+y^2=9$

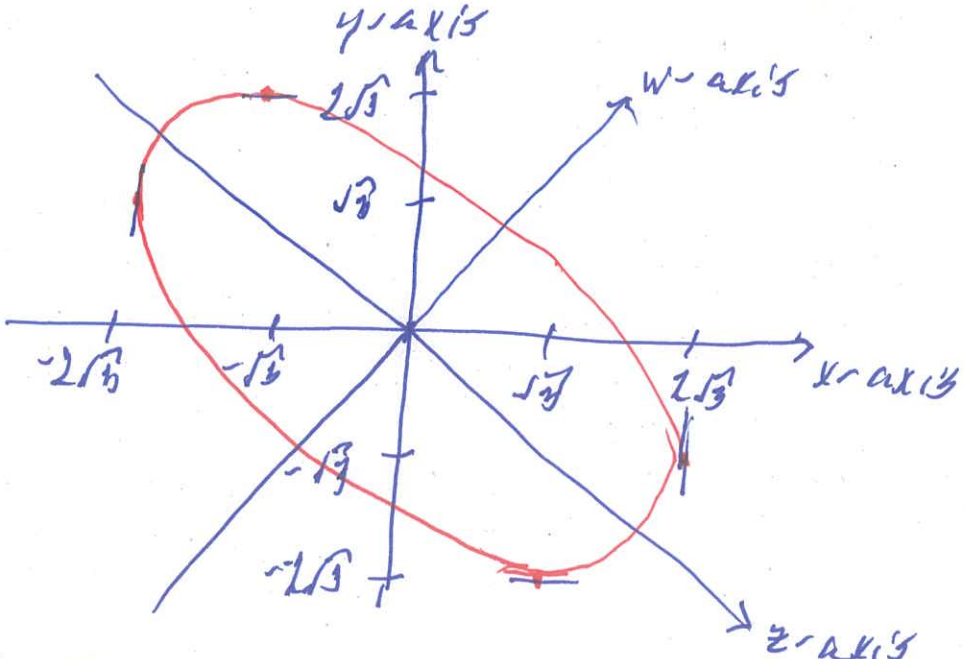
then $y = \frac{-1}{2}x$ and $x^2 + x\cdot\left(\frac{-1}{2}x\right) + \left(\frac{-1}{2}x\right)^2 = 9$

and $x^2 - \frac{1}{2}x^2 + \frac{1}{4}x^2 = 9$ so that $\frac{3}{4}x^2 = 9$.

So $x^2 = \frac{9\cdot 4}{3} = 12$. So $x = \sqrt{12} = 2\sqrt{3}$ or $x = -\sqrt{12} = -2\sqrt{3}$.

If $x = 2\sqrt{3}$ then $y = \frac{-1}{2}\cdot 2\sqrt{3} = -\sqrt{3}$

If $x = -2\sqrt{3}$ then $y = \frac{-1}{2}\cdot(-2\sqrt{3}) = \sqrt{3}$



Let $x = z + w$, $y = z - w$. Then $z = \frac{1}{2}(x+y)$, $w = \frac{1}{2}(x-y)$

$$\begin{aligned}9 = x^2 + xy + y^2 &= (z+w)^2 + (z+w)(z-w) + (z-w)^2 \\ &= z^2 + 2zw + w^2 \\ &\quad + z^2 \qquad - w^2 \\ &\quad + z^2 - 2zw + w^2 \\ &= 3z^2 + w^2\end{aligned}$$

So $9 = 3z^2 + w^2$. If $w = 0$ then $z^2 = 3$

If $z = 0$ then $w^2 = 1$.

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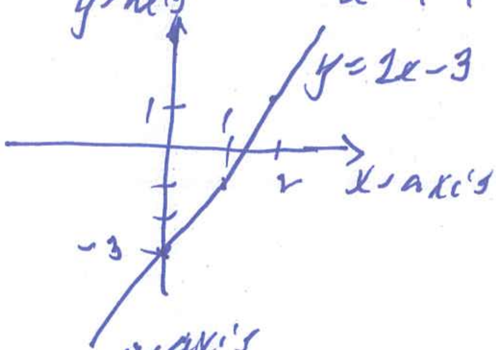
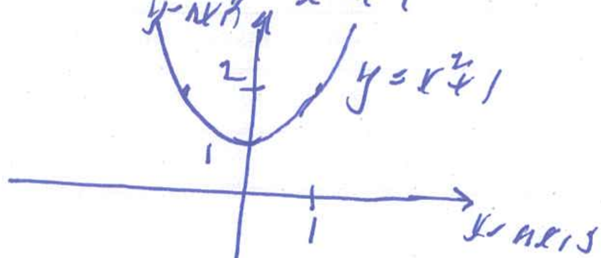
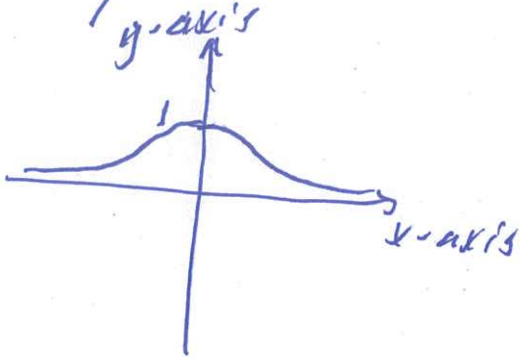
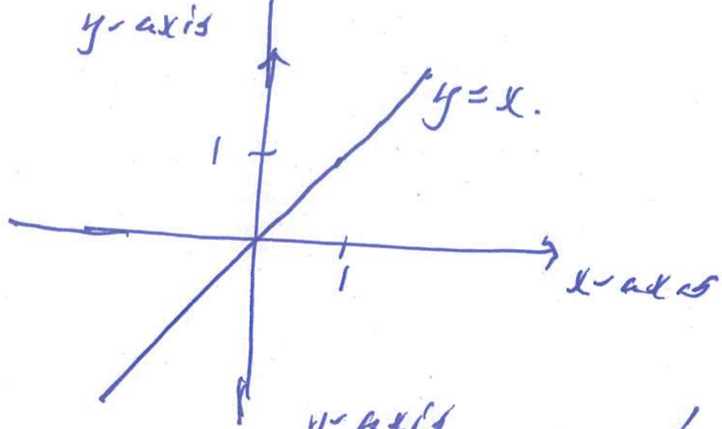
Example Graph $y = f(x)$ when

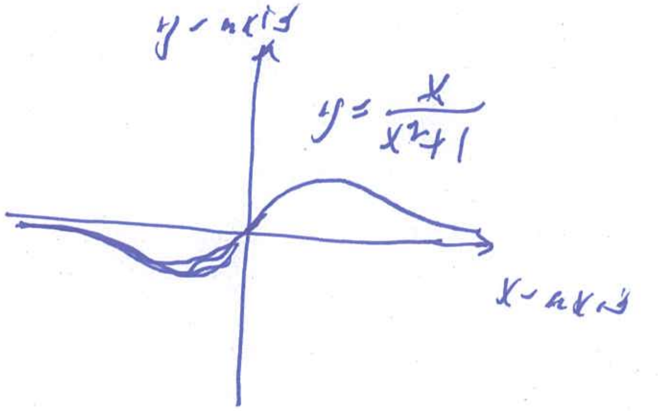
$$f(x) = \frac{2x^3 - 3x^2 - 3x - 3}{x^2+1} = 2x - 3 + \frac{x}{x^2+1}$$

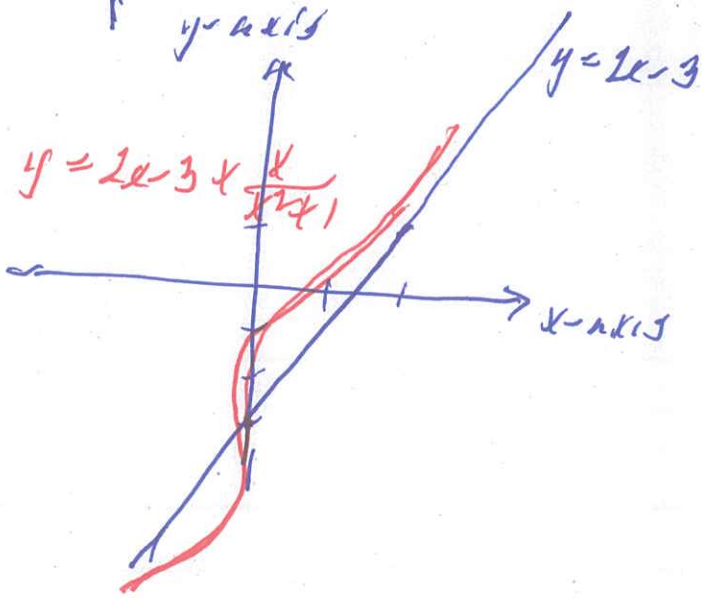












$$f'(x) = 2 + x(-1)(x^2+1)^{-2} \cdot 2x + (x^2+1)^{-1}$$

$$= 2 + \frac{-2x^2}{(x^2+1)^2} + \frac{1}{x^2+1} = \frac{2(x^2+1)^2 - 2x^2 + x^2 + 1}{(x^2+1)^2}$$

$$= \frac{2x^4 + 4x^2 + 2 - 2x^2 + x^2 + 1}{(x^2+1)^2} = \frac{2x^4 + 3x^2 - 2x + 3}{(x^2+1)^2}$$

So $f'(0) = 3$