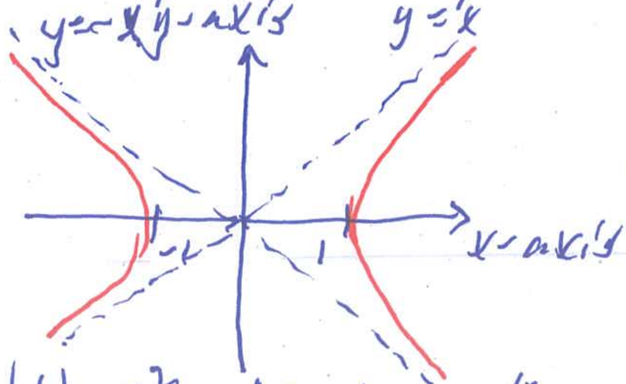
Asymptotes $f: \mathbb{R} \to \mathbb{R}$

An asymptote of the graph of $y=f(x)$ is another graph $y=g(x)$ such that the graph of $y=f(x)$ gets closer and closer to $y=g(x)$ as x gets closer and closer to a.

Example Graph $x^2-y^2=1$. (i.e. $\{(x,y)\in\mathbb{R}^2 \mid x^2-y^2=1\}$.



Notes

(a) If $y=0$ then $x^2=1$ and $x\in\{1,-1\}$.

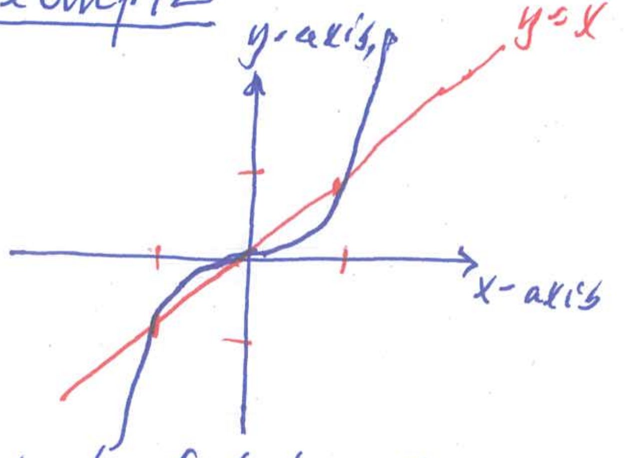
(b) $x^2-y^2=1$ is the same as $1-\frac{y^2}{x^2}=\frac{1}{x^2}$.

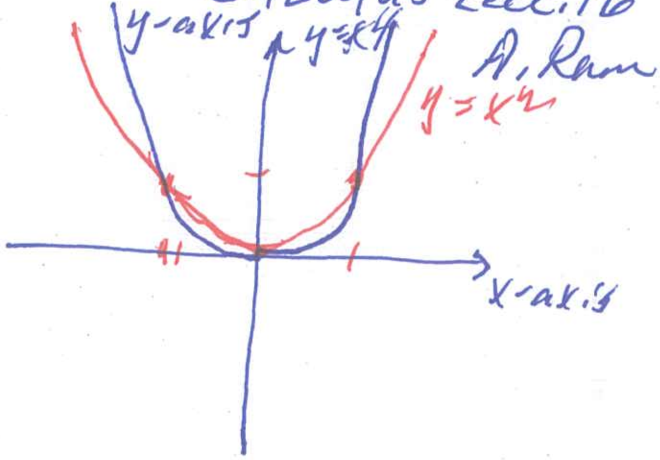
As $x\to\infty$ this becomes $1-\frac{y^2}{x^2}=0$. So $\frac{y^2}{x^2}=1$ and $y^2=x^2$ so that $y=\pm x$.

A stationary point of $y=f(x)$ is $(a,f(a))$ such that $f'(a)=0$.

A point of inflection of $y=f(x)$ is $(a,f(a))$ where the graph changes from concave up to concave down or from concave down to concave up.

Example





Let $f_n(x) = x^n$.

$f_1(x) = x$, $f_1'(x) = 1$, $f_1''(0) = 0$

$y = f_1(x)$ has no concavity, no points of inflection

$f_2(x) = x^2$, $f_2'(x) = 2x$, $f_2''(x) = 2$

$y = f_2(x)$ has no points of inflection.

$x = 0$ is a stationary point.

$f_3(x) = x^3$, $f_3'(x) = 3x^2$, $f_3''(x) = 6x$.

$y = f_3(x)$ has a point of inflection at $x = 0$.

and is increasing for $x \neq 0$
and has a stationary point at $x = 0$.

$f_4(x) = x^4$, $f_4'(x) = 4x^3$, $f_4''(x) = 12x^2$.

$y = f_4(x)$ does not have a point of inflection at $x = 0$ even though $f_4''(x) = 0$.

$y = f_3(x)$ does not have a minimum or maximum at $x = 0$ even though $f_3'(0) = 0$.

Example Graph $f(x) = \tan(x)$.

Let $y = \tan(x)$. Then

$$\frac{dy}{dx} = \frac{d\left(\frac{\sin(x)}{\cos(x)}\right)}{dx} = \frac{d(\sin(x)\cos(x)^{-1})}{dx}$$

$$= \sin(x)\frac{d\cos(x)^{-1}}{dx} + \frac{d\sin(x)}{dx}\cos(x)^{-1}$$

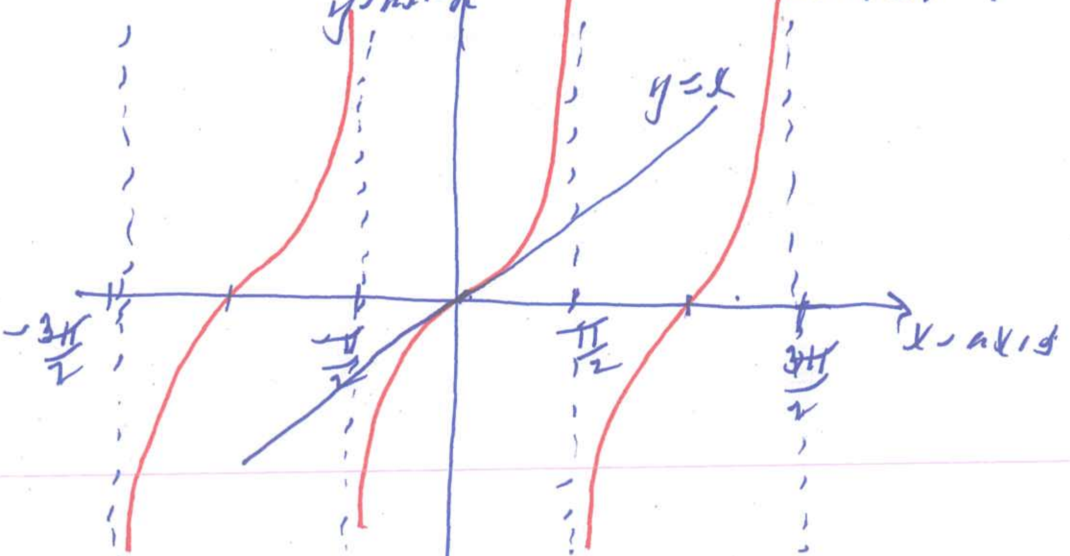
$$= \sin(x)(-1)\cos(x)^{-2}\frac{d\cos(x)}{dx} + \cos(x)\cos(x)^{-1}$$

$$= \frac{-\sin(x)}{\cos(x)^2}(-\sin(x)) + 1 = \frac{\sin(x)^2}{\cos(x)^2} + 1$$

$$= \frac{\sin(x)^2 + \cos(x)^2}{\cos(x)^2} = \frac{1}{\cos(x)^2} = \sec(x)^2.$$

$$\frac{d^2y}{dx^2} = \frac{d\sec(x)^2}{dx} = \frac{d\cos(x)^{-2}}{dx} = -2\cos(x)^{-3}\frac{d\cos(x)}{dx}$$

$$= \frac{-2}{\cos(x)^3}(-\sin(x)) = \frac{2\sin(x)}{\cos(x)^3}.$$



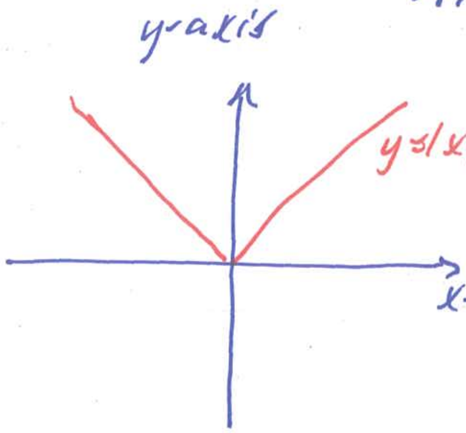
$y = \tan(x)$ has asymptote $x = \frac{\pi}{2}$ as $x \to \frac{\pi}{2}$.

$\left.\frac{dy}{dx}\right]_{x=0} = \frac{1}{\cos(0)^2} = \frac{1}{1^2} = 1$ and $y = x$ is the tangent line at $x=0$

Graph

Example $y=f(x)$ where $f(x)=|x|$. 15.04.2026 Calculus Lect. 16 A. Ram (4)

$$f(x)=|x|=\begin{cases} x, & \text{if } x\geq 0,\\ -x, & \text{if } x<0.\end{cases}$$



$$f'(x)=\begin{cases} 1, & \text{if } x>0\\ -1, & \text{if } x<0\end{cases}$$

$$f''(x)=\begin{cases} 0, & \text{if } x>0\\ 0, & \text{if } x<0\end{cases}$$ and $f''(0)=0$.

Example Find horizontal and vertical tangent lines of $x^2+xy+y^2=9$. Take derivative of both sides.

$$2x+x\frac{dy}{dx}+y+2y\frac{dy}{dx}=0. \text{ So } (x+2y)\frac{dy}{dx}=-y-2x.$$

So $$\frac{dy}{dx}=\frac{-y-2x}{x+2y}$$

Case 1: If $-y-2x$ is 0 then $\frac{dy}{dx}$ is 0 and we should get a slope 0 tangent line.

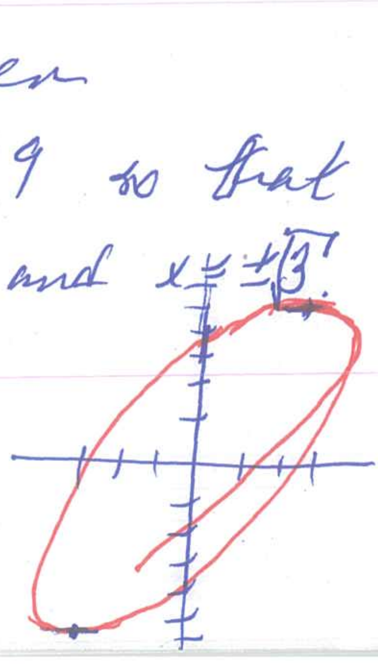
If $-y-2x=0$ and $x^2+xy+y^2=9$ then

$y=-2x$ and $x^2+x(-2x)+(-2x)^2=9$ so that

$x^2-2x^2+4x^2=9$ and $3x^2=9$ and $x=\pm\sqrt{3}$.

So, if $x=\sqrt{3}$ then $y=-2\sqrt{3}$

if $x=-3$ then $y=-2\cdot(-\sqrt{3})=2\sqrt{3}$



<u>Case 2</u>: If $x+2y$ is 0 then $\frac{dy}{dx}$ is probably infinite and we should get a slope ∞ (vertical) tangent line. If $x+2y=0$ and $x^2+xy+y^2=9$ then

$y=-\frac{1}{2}x$ and $x^2+x\left(-\frac{1}{2}x\right)+\left(-\frac{1}{2}x\right)^2=9$ and

$x^2-\frac{1}{2}x^2+\frac{1}{4}x^2=9$ so that $\frac{3}{4}x^2=9$ and $x=\pm\sqrt{12}$.
$=\pm 2\sqrt{3}$.

If $x=2\sqrt{3}$ then $y=-\frac{1}{2}\cdot 2\sqrt{3}=-\sqrt{3}$

If $x=-2\sqrt{3}$ then $y=-\frac{1}{2}(-2\sqrt{3})=\sqrt{3}$.