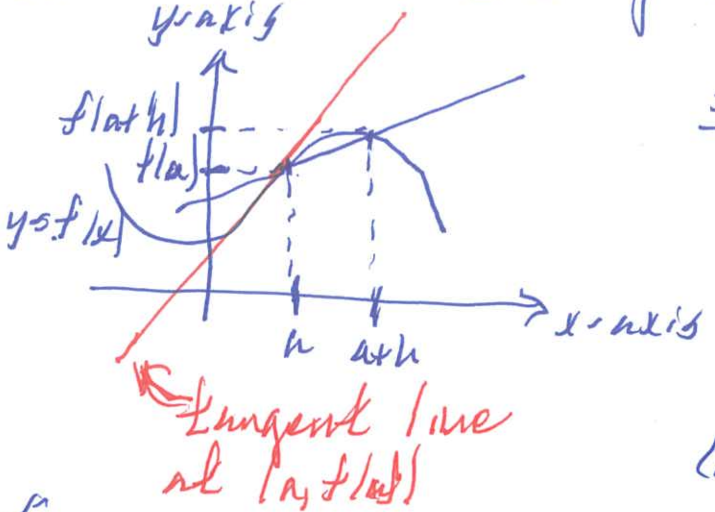
15.04.2026 ①
Calculus Lect 15
A. Ram

Does anything ever change?

Think about

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

in terms of the graph



$$\frac{f(a+h)-f(a)}{h} = \frac{\text{change in } f}{\text{change in } a}$$

$$= \frac{\text{rise}}{\text{run}}$$

= slope of line connecting $(a, f(a))$ and $(a+h, f(a+h))$.

So $f'(a)$ = slope of the graph of f at $(a, f(a))$

The <u>tangent line to $y=f(x)$ at $x=a$</u> is the line of slope $f'(a)$ going through $(a, f(a))$

$$y - f(a) = f'(a)(x-a)$$

(equation of tangent line at $(a, f(a))$)

Fundamental theorem of change

If $f:\mathbb{R}\to\mathbb{R}$ is a nice function

then $f'(a) = \left.\frac{df}{dx}\right|_{x=a}$

where $\left.\frac{df}{dx}\right|_{x=a}$ is $\frac{df}{dx}$ evaluated at $x=a$.

Problem 3.6 (1) Graph $y = f(x)$ where Calculus Lect. 15

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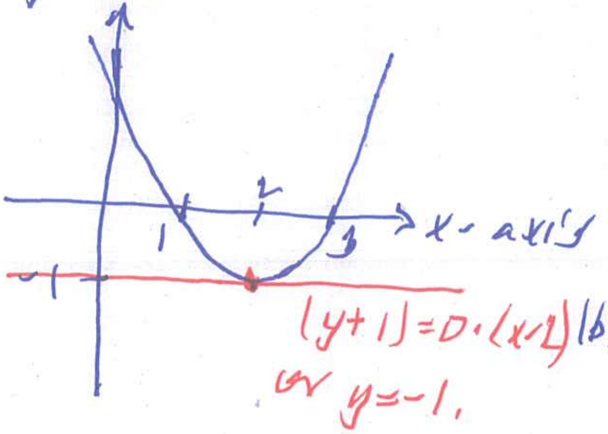
y-axis $f(x) = x^2 - 4x + 3$



Notes:

(a) Since $x^2 - 4x + 3 = (x-1)(x-3)$ then $f(3) = 0$ and $f(1) = 0$.

(b) $y = x^2 - 4x + 3$ is a concave up parabola.

(c) $$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{\begin{pmatrix} (a+h)^2 - 4(a+h) + 3 \\ -(a^2 - 4a + 3) \end{pmatrix}}{h}$$

$$= \lim_{h \to 0} \frac{\begin{pmatrix} a^2 + 2ah + h^2 - 4a - 4h + 3 \\ -a^2 \qquad + 4a \qquad - 3 \end{pmatrix}}{h} = \lim_{h \to 0} \frac{(2ah + h^2 - 4h)}{h}$$

$$= \lim_{h \to 0} (2a + h - 4) = 2a - 4 = 2(a-2)$$

If $a = 2$ then $f'(a) = 0$ and $f(2) = 2^2 - 4 \cdot 2 + 3 = -1$.

So the tangent line at $(2, -1)$ has slope 0.

(d) $f'(a) = 2(a-2) < 0$ if $a < 2$.

$f'(a) = 2(a-2) > 0$ if $a > 2$.

So $y = f(x)$ is decreasing if $a < 2$.

$y = f(x)$ is increasing if $a > 2$.

A function $f:\mathbb{R}\to\mathbb{R}$ is increasing at $x=a$

if it is going up at $x=a$,

i.e. if $f(a+h)>f(a)$ for all small $h>0$

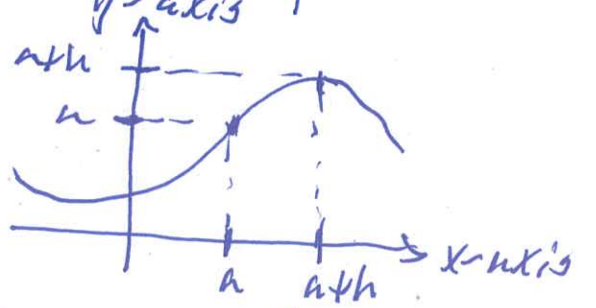
i.e. if the slope $f'(a)>0$.

A function $f:\mathbb{R}\to\mathbb{R}$ is decreasing at $x=a$

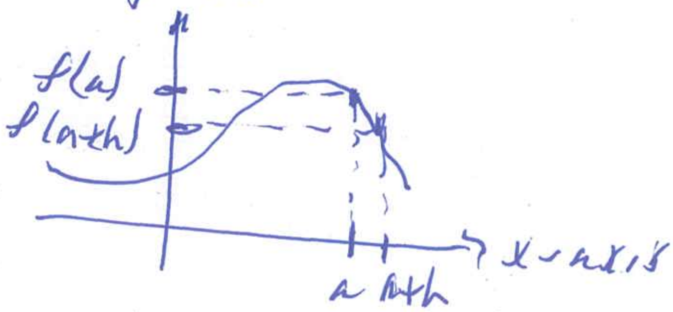
if it is going down at $x=a$,

i.e. if $f(a+h)<f(a)$ for all small $h<0$

i.e. if the slope $f'(a)<0$.



increasing at $x=a$



decreasing at $x=a$

$f:\mathbb{R}\to\mathbb{R}$ is concave up at $x=a$

if it is right side up bowl shaped at $x=a$

i.e. if the slope is getting larger at $x=a$

i.e. if $f'(x)$ is increasing at $x=a$

i.e. if $f''(a)>0$,

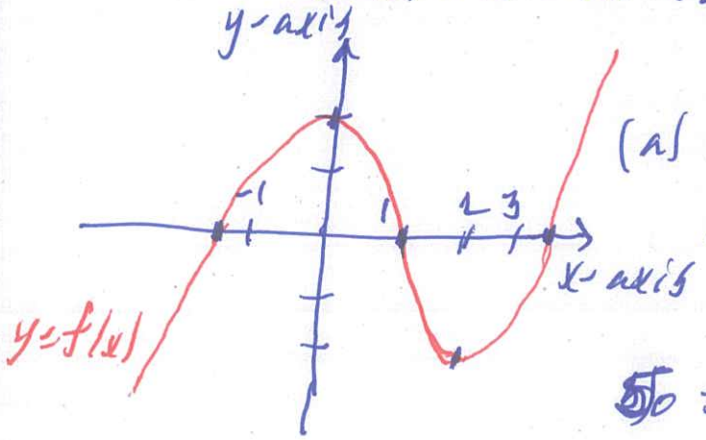
where $f''(a)=\lim\limits_{h\to 0}\dfrac{f'(a+h)-f'(a)}{h}$

f is concave down at $x=a$
 if it is upside down bowl shaped at $x=a$
i.e. if the slope of f is getting smaller at $x=a$
i.e. if $f'(x)$ is decreasing at $x=a$
i.e. if $f''(a)<0$

A point of inflection is a point where f changes from concave up to concave down or from concave down to concave up.

Problem 3.6(2) Graph $y=f(x)$ where

$$f(x)=x^3-3x^2+2=(x-1)(x-(1+\sqrt{3}))(x-(1-\sqrt{3}))$$



Notes:

(a) $x^3-3x^2+2=(x-1)(x^2-2x-2)$

and $\dfrac{-(-2)\pm\sqrt{(-2)^2-4(-2)}}{2}=1\pm\sqrt{3}$

So $f(1)=0$, $f(1+\sqrt{3})=0$, $f(1-\sqrt{3})=0$

(b) As $x\to\infty$ then $x^3-3x^2+2\to\infty$
 As $x\to-\infty$ then $x^3-3x^2+2\to-\infty$.

(c) Using the fundamental theorem of change

$$f'(a)=\frac{df}{dx}\Big]_{x=a}=(3x^2-6x)\Big]_{x=a}=3(a^2-6a)=3a(a-2).$$

So the slope of $y=f(x)$ is 0 if $a=2$ or $a=0$
 and $f(2)=8-3\cdot4+2=-2$ and $f(0)=2$

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(d) Since $f'(x) = 3x^2 - 6x$
then the fundamental theorem of change gives

$$f''(x) = (6x-6)\Big]_{x=a} = 6a - 6 = 6(a-1)$$

So f is concave up for $a > 1$
f is concave down for $a < 1$
and f has a point of inflection at $a = 1$.