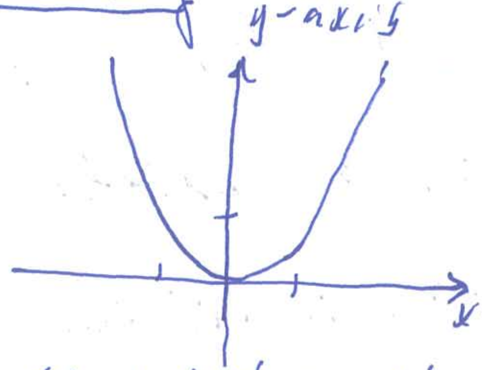
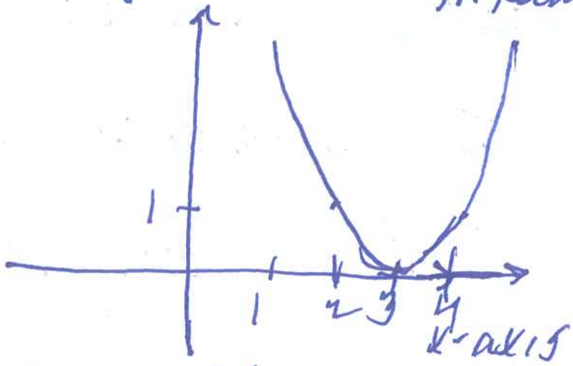
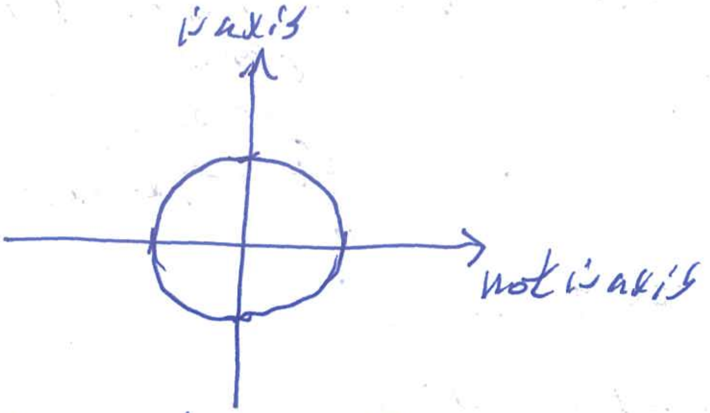
Shifting



$\{(x,y)\in\mathbb{R}^2 \mid y = x^2\}$



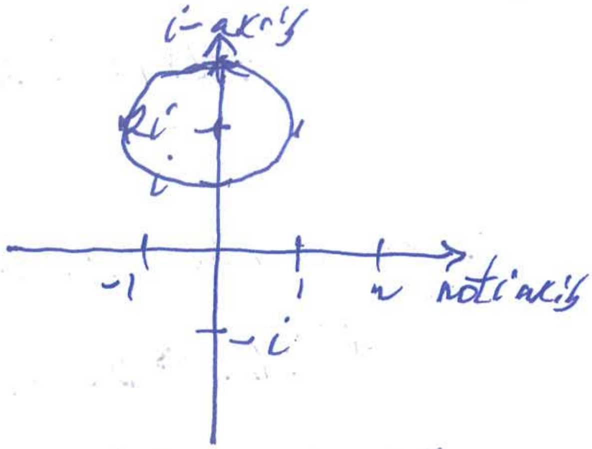
$\{(x,y)\in\mathbb{R}^2 \mid y = (x-3)^2\}$



$\{z\in\mathbb{C} \mid |z| = 1\}$

$= \left\{z\in\mathbb{C} \,\middle|\, \begin{array}{l}\text{distance of } z\\ \text{to } 0 \text{ is } 1\end{array}\right\}$

$= \left(\begin{array}{l}\text{circle of radius } 1\\ \text{centred at } 0\end{array}\right)$

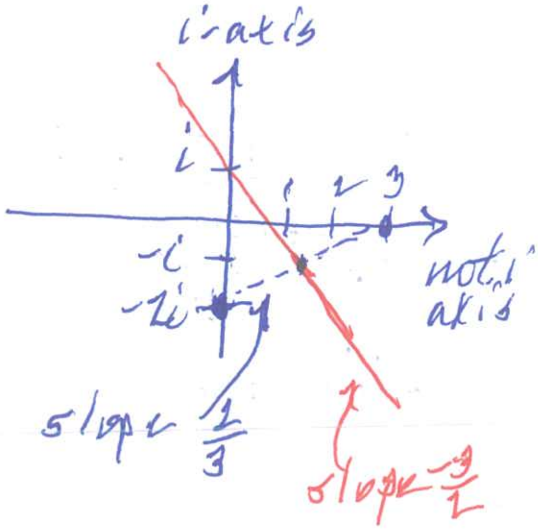


$\{z\in\mathbb{C} \mid |z-2i| = 1\}$

$= \left\{z\in\mathbb{C} \,\middle|\, \begin{array}{l}\text{distance of } z\\ \text{to } 2i \text{ is } 1\end{array}\right\}$

$= \left(\begin{array}{l}\text{circle of radius } 1\\ \text{centered at } 2i\end{array}\right)$

Problem 2.13



$\{z\in\mathbb{C} \mid |z-3| = |z+2i|\}$

$= \left\{z\in\mathbb{C} \,\middle|\, \begin{array}{l}\text{distance } z \text{ to } 3\\ \text{equals}\\ \text{distance } z \text{ to } -2i\end{array}\right\}$

$= \left\{z\in\mathbb{C} \,\middle|\, \begin{array}{l}z \text{ is on the line}\\ \text{halfway between } 3 \text{ and}\\ -2i\end{array}\right\}$

$= \left\{z\in\mathbb{C} \,\middle|\, \begin{array}{l}z \text{ is on the line}\\ \text{with slope } -3/2\\ \text{through } 1.5-i\end{array}\right\}$

So

$$\{z \in \mathbb{C} \mid |z-3| = |z+2i|\}$$
$$= \{x+yi \mid y+1 = -\tfrac{3}{2}(x - \tfrac{3}{2})\}$$
$$= \{x+yi \mid y = -\tfrac{3}{2}x + \tfrac{9}{4} - 1\}$$
$$= \{x+yi \mid y = -\tfrac{3}{2}x + \tfrac{5}{4}\}.$$

<u>The derivation</u> $J \xrightarrow{\frac{d}{dx}} \frac{dJ}{dx}.$

(1) (normalization) $\frac{dx}{dx} = 1$.

(2) (linearity) If $c_1, c_2 \in \mathbb{C}$ then

$$\frac{d(c_1 f + c_2 g)}{dx} = c_1 \frac{df}{dx} + c_2 \frac{dg}{dx}$$

(3) (Product rule) $\frac{d(fg)}{dx} = f\frac{dg}{dx} + \frac{df}{dx} \cdot g$

We proved (0) $\frac{d1}{dx} = 0$

(a) If $n \in \mathbb{Z}_{>0}$ then $\frac{dJ^n}{dx} = nJ^{n-1}\frac{dJ}{dx}$

(b) $\frac{de^J}{dx} = e^J \frac{dJ}{dx}$ and $\frac{d\cos(J)}{dx} = -\sin(J)\frac{dJ}{dx}$

(c) If $n \in \mathbb{Z}_{>0}$ then $\frac{dJ^{-n}}{dx} = -nJ^{-n-1}\frac{dJ}{dx}$

(d) If $c \in \mathbb{C}$ then $\frac{dc}{dx} = \frac{d(c \cdot 1)}{dx} = c\frac{d1}{dx} = c \cdot 0 = 0.$

Problem 3.2 (1). Let $y = (x^2+1)(x-3)$. Find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{d}{dx}\left((x^2+1)\cdot(x-3)\right) = (x^2+1)\frac{d(x-3)}{dx} + \frac{d(x^2+1)}{dx}\cdot(x-3)$$

$$= (x^2+1)\left(\frac{dx}{dx} - \frac{d3}{dx}\right) + \left(\frac{dx^2}{dx} + \frac{d1}{dx}\right)\cdot(x-3)$$

$$= (x^2+1)(1-0) + \left(2x\frac{dx}{dx} + 0\right)(x-3)$$

$$= x^2+1-2x(x-3) = x^2+1-2x^2+6x$$

$$= -x^2+6x+1$$

Problem 3.2 (2). Let $y = \frac{x^2+1}{x-1}$. Find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2+1}{x-1}\right) = \frac{d}{dx}\left((x^2+1)(x-1)^{-1}\right)$$

$$= (x^2+1)\frac{d((x-1)^{-1})}{dx} + \frac{d(x^2+1)}{dx}(x-1)^{-1}$$

$$= (x^2+1)(-1)(x-1)^{-2}\frac{d(x+1)}{dx} + (2x+0)\frac{1}{x-1}.$$

$$= \frac{-(x^2+1)}{(x-1)^2}(1+0) + \frac{2x}{x-1}$$

$$= \frac{-x^2-1}{(x-1)^2} + \frac{2x(x-1)}{(x-1)^2} = \frac{-x^2-1+2x^2-2x}{(x-1)^2}$$

$$= \frac{x^2-2x-1}{(x-1)^2}.$$

Problem 3.1 (3) Let $y=e^{x^2}$. Find $\frac{dy}{dx}$.

Since $\frac{de^J}{dx} = e^J\frac{dJ}{dx}$ then

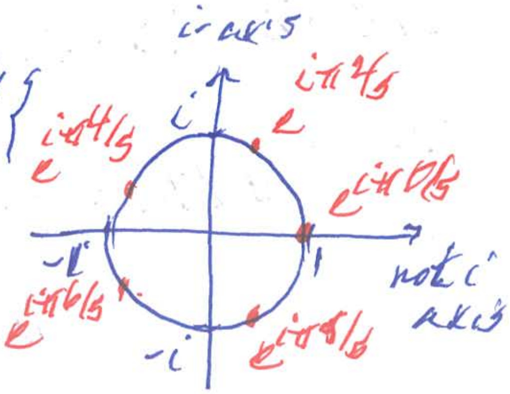
$$\frac{dy}{dx} = \frac{de^{x^2}}{dx} = e^{x^2}\frac{dx^2}{dx} = e^{x^2}\cdot 2x\frac{dx}{dx} = 2xe^{x^2}\cdot 1 = 2xe^{x^2}$$

Problem 1.23 (4)

$$\{y\in\mathbb{C} \mid y^5=1\}$$

$$=\{e^{2\pi i\frac{0}{5}}, e^{i\pi\frac{2}{5}}, e^{i\pi\frac{4}{5}}, e^{i\pi 6/5}, e^{i\pi 8/5}\}$$

= {5 equally spaced points on a circle of radius 1 including 1}



$$\{z\in\mathbb{C}\mid z^5 = 32e^{5\pi i/3}\}$$

$$=\{z\in\mathbb{C}\mid z^5 = 2^5(e^{i\pi/3})^5\}$$

$$=\{z\in\mathbb{C}\mid \frac{z^5}{2^5(e^{i\pi/3})^5}=1\}$$

$$=\{z\in\mathbb{C}\mid \left(\frac{z}{2e^{i\pi/3}}\right)^5=1\}$$

(substitute $y=\frac{z}{2e^{i\pi/3}}$ so $z=2e^{i\pi/3}y$)

$$=\{2e^{i\pi/3}y\mid y^5=1\}$$

$$=\{2e^{i\pi/3}e^{i\pi 0/5}, 2e^{i\pi/3}e^{i\pi\frac{2}{5}}, 2e^{i\pi/3}e^{i\pi 4/5}, 2e^{i\pi/3}e^{i\pi 6/5}, 2e^{i\pi/3}e^{i\pi 8/5}\}$$

$$=\{2e^{i\pi(\frac{1}{3})}, 2e^{i\pi(\frac{1}{3}+\frac{2}{5})}, 2e^{i\pi(\frac{1}{3}+\frac{4}{5})}, 2e^{i\pi(\frac{1}{3}+\frac{6}{5})}, 2e^{i\pi(\frac{1}{3}+\frac{8}{5})}\}.$$

Problem Show that $\frac{dJ^{-1}}{dx} = -J^{-2}\frac{dJ}{dx}$ Calculuslect.12
A.Ram

Since
$$0 = \frac{d1}{dx} = \frac{d(J \cdot J^{-1})}{dx} = J\frac{dJ^{-1}}{dx} + \frac{dJ}{dx}J^{-1}$$
then
$$J\frac{dJ^{-1}}{dx} = -\frac{dJ}{dx}J^{-1}.$$
So
$$\frac{dJ^{-1}}{dx} = -J^{-2}\frac{dJ}{dx}.$$

Problem Show that if $n \in \mathbb{Z}_{>0}$ then
$$\frac{dJ^{-n}}{dx} = -nJ^{-n-1}\frac{dJ}{dx}.$$

Since
$$0 = \frac{d1}{dx} = \frac{d(J^n \cdot J^{-n})}{dx} = J^n\frac{dJ^{-n}}{dx} + \frac{dJ^n}{dx}J^{-n}$$
then
$$J^n\frac{dJ^{-n}}{dx} = -J^{-n}\frac{dJ^n}{dx}.$$
So
$$\frac{dJ^{-n}}{dx} = -J^{-2n}nJ^{n-1}\frac{dJ}{dx} = -nJ^{-n-1}\frac{dJ}{dx}.$$

Problem 3.2 (4)

Let $y = \log(3x^2+1)$. Find $\frac{dy}{dx}$, ~~and~~

Since
$y = \log(3x^2+1)$ then $e^y = 3x^2+1$.

So
$$\frac{de^y}{dx} = \frac{d(3x^2+1)}{dx} = 3\frac{dx^2}{dx} + \frac{d1}{dx} = 3\cdot 2x\frac{dx}{dx} + 0 = 6x$$

So
$e^y\frac{dy}{dx} = 6x$. So $\frac{dy}{dx} = \frac{1}{e^y}\cdot 6x = \frac{6x}{3x^2+1}$. //

Problem 3.2 (6). Let $y = \sqrt{x}$. Find $\frac{dy}{dx}$.

Since $y = \sqrt{x}$ then $y^2 = x$.

So $\frac{dy^2}{dx} = \frac{dx}{dx} = 1$. So $2y\frac{dy}{dx} = 1$.

So $\frac{dy}{dx} = \frac{1}{2y} = \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-1/2}$. //

Problem 3.2 (10) Let $y = \frac{\log(x)}{x}$. Find $\frac{dy}{dx}$.

Since $y = \frac{\log(x)}{x}$ then $xy = \log(x)$ and $e^{xy} = x$.

So $\frac{de^{xy}}{dx} = \frac{dx}{dx}$. So $e^{xy}\frac{d(xy)}{dx} = 1$.

So $\frac{d(xy)}{dx} = \frac{1}{e^{xy}} = \frac{1}{x}$. So $x\frac{dy}{dx} + \frac{dx}{dx}y = \frac{1}{x}$.

So $x\frac{dy}{dx}+y=\frac{1}{x}$. So $x\frac{dy}{dx}=\frac{1}{x}-y$.

So $\frac{dy}{dx}=\frac{1}{x}\left(\frac{1}{x}-y\right)=\frac{1}{x^2}-\frac{y}{x}=\frac{1}{x^2}-\frac{xy}{x^2}$

$=\frac{1-\log|x|}{x^2}$. //

Problem 3.10 (1) Suppose $x^2-xy+y^3=8$. Find $\frac{dy}{dx}$

$$\frac{d(x^2-xy+y^3)}{dx}=\frac{d8}{dx}.$$

So $\frac{dx^2}{dx}-\frac{d(xy)}{dx}+\frac{dy^3}{dx}=0$.

So $2x\frac{dx}{dx}-\left(x\frac{dy}{dx}+\frac{dx}{dx}y\right)+3y^2\frac{dy}{dx}=0$.

So $2x-x\frac{dy}{dx}+y+3y^2\frac{dy}{dx}=0$.

So $(-x+3y^2)\frac{dy}{dx}=-2x-y$

So $\frac{dy}{dx}=\frac{-2x-y}{-x+3y^2}$