30.03.2026 ①
Calculus Lect. 12
A. Ram

The derivation $\frac{d}{dx}$

expressions $f \to$ 

$\frac{df}{dx}$ expressions

satisfies

(1) (normalization) $\frac{dx}{dx} = 1$

(2) (Linearity) If $c_1, c_2 \in \mathbb{C}$ then
$$\frac{d(c_1 f + c_2 g)}{dx} = c_1 \frac{df}{dx} + c_2 \frac{dg}{dx}$$

(3) (Product rule)
$$\frac{d(fg)}{dx} = f \frac{dg}{dx} + \frac{df}{dx} g.$$

<u>Problem</u> Prove that $\frac{d1}{dx} = 0.$

<u>Proof</u>
$$\frac{d1}{dx} = \frac{d(1 \cdot 1)}{dx} = 1 \frac{d1}{dx} + \frac{d1}{dx} \cdot 1 = \frac{d1}{dx} + \frac{d1}{dx}.$$

Subtract $\frac{d1}{dx}$ from both sides to get
$$0 = \frac{d1}{dx}. \quad //.$$

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Calculus Lect. 12 (2)

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<u>Problem</u> Prove that $\frac{dJ^2}{dx} = 2J\frac{dJ}{dx}$

<u>Proof</u>

$$\frac{dJ^2}{dx} = \frac{d(J\cdot J)}{dx} = J\frac{dJ}{dx} + \frac{dJ}{dx}\cdot J = 2J\frac{dJ}{dx}.$$

<u>Problem</u> Prove that $\frac{dJ^3}{dx} = 3J^2\frac{dJ}{dx}$.

<u>Proof</u>

$$\frac{dJ^3}{dx} = \frac{d(J^2\cdot J)}{dx} = J^2\frac{dJ}{dx} + \frac{dJ^2}{dx}\cdot J$$

$$= J^2\frac{dJ}{dx} + 2J\frac{dJ}{dx}J = (J^2 + 2J^2)\frac{dJ}{dx}$$

$$= 3J^2\frac{dJ}{dx}.$$

<u>Problem</u> Prove that if $n \in \mathbb{Z}_{>0}$ then

$$\frac{dJ^n}{dx} = nJ^{n-1}\frac{dJ}{dx}.$$

<u>Proof</u> The proof is by induction on n.

The base case is $\frac{dJ^2}{dx} = 2J\frac{dJ}{dx}$ proved above.

Induction step: Assume $\frac{dJ^n}{dx} = nJ^{n-1}\frac{dJ}{dx}$

To show: $\frac{d(J^{n+1})}{dx} = (n+1)J^{n+1-1}\frac{dJ}{dx}.$

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Calculus Lect. 12 (3)

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$$\frac{d(J^{n+1})}{dx} = \frac{d(J^n \cdot J)}{dx} = J^n \cdot \frac{dJ}{dx} + \frac{dJ^n}{dx} \cdot J$$

$$= J^n \frac{dJ}{dx} + nJ^{n-1} \frac{dJ}{dx} \cdot J$$

$$= (J^n + nJ^n) \frac{dJ}{dx} = (n+1) J^n \frac{dJ}{dx}$$

$$= (n+1) J^{n+1-1} \frac{dJ}{dx}. \quad //$$

<u>Problem</u> Prove that $\frac{de^J}{dx} = e^J \frac{dJ}{dx}$.

<u>Proof</u>

$$\frac{de^J}{dx} = \frac{d\left(1 + J + \frac{1}{2!}J^2 + \frac{1}{3!}J^3 + \frac{1}{4!}J^4 + \cdots\right)}{dx}$$

$$= \frac{d1}{dx} + \frac{dJ}{dx} + \frac{1}{2!}\frac{dJ^2}{dx} + \frac{1}{3!}\frac{dJ^3}{dx} + \frac{1}{4!}\frac{dJ^4}{dx} + \cdots$$

$$= 0 + \frac{dJ}{dx} + \frac{1}{2!} 2J \frac{dJ}{dx} + \frac{1}{3!} 3J^2 \frac{dJ}{dx} + \frac{1}{4!} 4J^3 \frac{dJ}{dx} + \cdots$$

$$= \frac{dJ}{dx}\left(1 + J + \frac{1}{2!}J^2 + \frac{1}{3!}J^3 + \cdots\right)$$

$$= \frac{dJ}{dx} e^J. \quad //$$

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Calculus Lect 12 ④

A. Ram

Problem Prove that $\frac{d\cos(J)}{dx} = -\sin(J)\frac{dJ}{dx}$

Proof

$$\frac{d\cos(J)}{dx} = \frac{d\ \frac{1}{2}(e^{iJ}+e^{-iJ})}{dx}$$

$$= \frac{1}{2}\frac{d(e^{iJ})}{dx} + \frac{1}{2}\frac{d(e^{-iJ})}{dx}$$

$$= \frac{1}{2}e^{iJ}\frac{d(iJ)}{dx} + \frac{1}{2}e^{-iJ}\frac{d(-iJ)}{dx}$$

$$= \frac{1}{2}e^{iJ}i\frac{dJ}{dx} + \frac{1}{2}e^{-iJ}(-i)\frac{dJ}{dx}$$

$$= \frac{1}{2}i\frac{dJ}{dx}(e^{iJ}-e^{-iJ})$$

$$= \frac{1}{2}i\cdot\frac{i}{i}(e^{iJ}-e^{-iJ})\frac{dJ}{dx}$$

$$= -\sin(J)\frac{dJ}{dx}.$$ //.

Problem Prove that $\frac{d\sin(J)}{dx} = \cos(J)\frac{dJ}{dx}$

Do this in a similar way to the way we did $\frac{d\cos(J)}{dx} = -\sin(J)\frac{dJ}{dx}$