26.03.2026
Calculus Lect. 10 ①
A. Ram

Trigonometry

What is $\cos(4+3i)$?

Recall, $2! = 2\cdot 1$
$3! = 3\cdot 2\cdot 1$
$4! = 4\cdot 3\cdot 2\cdot 1$
$\vdots$

$$e^z = 1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \frac{1}{4!}z^4 + \dots$$

then

$$\cos(z) = \frac{1}{2}(e^{iz} + e^{-iz}) \text{ and } \sin(z) = \frac{1}{2i}(e^{iz} - e^{-iz}).$$

The 2nd graders dream (the most important theorem in math)

$$e^z e^w = e^{z+w}.$$

Problem Prove that $\cos(z) + i\sin(z) = e^{iz}$.

Proof

$$\begin{aligned}\cos(z) + i\sin(z) &= \frac{1}{2}(e^{iz} + e^{-iz}) + i\frac{1}{2i}(e^{iz} - e^{-iz}) \\ &= \frac{1}{2}e^{iz} + \frac{1}{2}e^{-iz} + \frac{1}{2}e^{iz} - \frac{1}{2}e^{-iz} \\ &= e^{iz}.\end{aligned}$$

$\blacksquare$

So

$$r\cos(z) + ir\sin(z) = re^{iz}.$$

What are $\cos(2z)$ and $\sin(2z)$?

$$\cos(2z)+i\sin(2z)=e^{i2z}=e^{i(z+z)}$$
$$=e^{iz}e^{iz}=(\cos(z)+i\sin(z))^2$$
$$=\cos(z)^2+2i\cos(z)\sin(z)+i^2\sin(z)^2$$
$$=(\cos(z)^2-\sin(z)^2)+i\,2\cos(z)\sin(z).$$

So
$$\cos(2z)=\cos(z)^2-\sin(z)^2 \quad \text{and}$$
$$\sin(2z)=2\cos(z)\sin(z).$$

What are $\cos(z+w)$ and $\sin(z+w)$?

$$\cos(z+w)+i\sin(z+w)=e^{i(z+w)}=e^{iz}e^{iw}$$
$$=(\cos(z)+i\sin(z))(\cos(w)+i\sin(w))$$
$$=\cos(z)\cos(w)+i\cos(z)\sin(w)$$
$$+i\sin(z)\cos(w)+i^2\sin(z)\sin(w)$$
$$=\cos(z)\cos(w)-\sin(z)\sin(w)$$
$$+i(\cos(z)\sin(w)+\sin(z)\cos(w))$$

So
$$\cos(z+w)=\cos(z)\cos(w)-\sin(z)\sin(w)$$
and $\sin(z+w)=\sin(z)\cos(w)+\sin(w)\cos(z).$

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Since $\cos(z) = \frac{1}{2}(e^{iz} + e^{-iz})$ then

$$\cos(-z) = \frac{1}{2}(e^{-iz} + e^{iz}) = \cos(z).$$

Since $\sin(z) = \frac{1}{2i}(e^{iz} - e^{-iz})$ then

$$\sin(-z) = \frac{1}{2i}(e^{-iz} - e^{iz}) = -\frac{1}{2i}(e^{iz} - e^{-iz})$$
$$= -\sin(z).$$

Problem Prove that $\cos(z)^2 + \sin(z)^2 = 1$.

Proof $1 = e^0 = e^{i \cdot 0} = e^{i(z+(-z))} = e^{iz}e^{-iz}$

$$= (\cos(z) + i\sin(z))(\cos(-z) + i\sin(-z))$$
$$= (\cos(z) + i\sin(z))(\cos(z) - i\sin(z))$$
$$= \cos(z)^2 - i^2\sin(z)^2$$
$$= \cos(z)^2 + \sin(z)^2. //$$

The 2nd grader's dream (the most important theorem in math)

$$e^{z+w} = e^z e^w.$$

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$$
\begin{aligned}
e^{z+w} &= 1 + (z+w) + \frac{1}{2!}(z+w)^2 + \frac{1}{3!}(z+w)^3 + \frac{1}{4!}(z+w)^4 + \cdots \\
&= 1 + z + w + \frac{1}{2}(z^2 + 2zw + w^2) + \frac{1}{3!}(z^3 + 3z^2w + 3zw^2 + w^3) + \frac{1}{4!}(z^4 + 4z^3w + 6z^2w^2 + 4zw^3 + w^4) + \cdots
\end{aligned}
$$

$$
\begin{aligned}
&= 1 \\
&\quad + z + w \\
&\quad + \frac{1}{2!}z^2 + wz + \frac{1}{2!}w^2 \\
&\quad + \frac{1}{3!}z^3 + \frac{1}{2!}wz^2 + z\frac{1}{2!}w^2 + \frac{1}{3!}w^3 \\
&\quad + \frac{1}{4!}z^4 + \frac{1}{3!}z^3w + \frac{1}{2!}z^2\frac{1}{2!}w^2 + z\frac{1}{3!}w^3 + \frac{1}{4!}w^4 \\
&\quad + \cdots
\end{aligned}
$$

$$= e^z + e^z w + e^z\frac{1}{2!}w^2 + e^z\frac{1}{3!}w^3 + \cdots$$

$$= e^z\left(1 + w + \frac{1}{2!}w^2 + \frac{1}{3!}w^3 + \cdots\right) = e^z e^w. \quad //$$