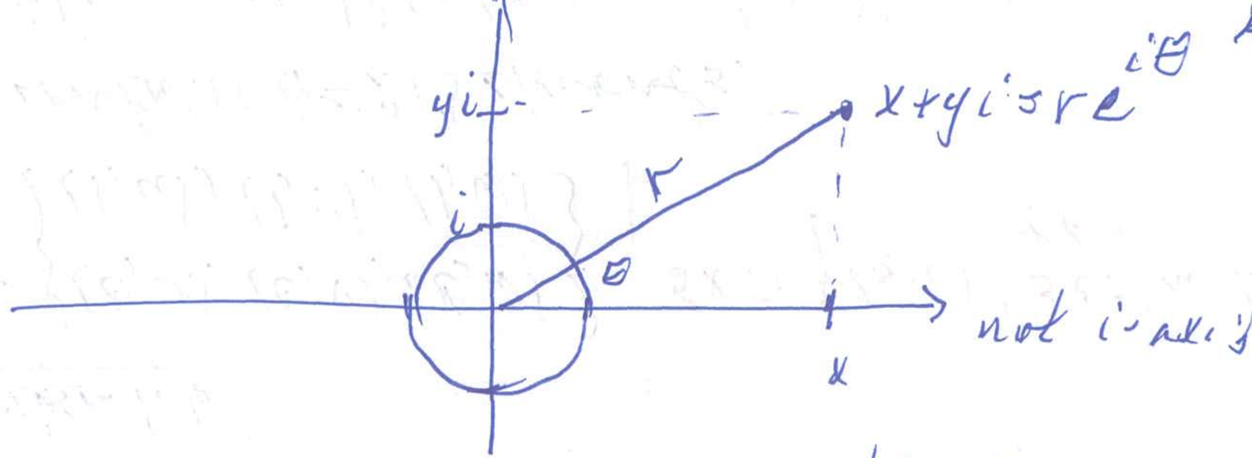
Calculus Lect 10

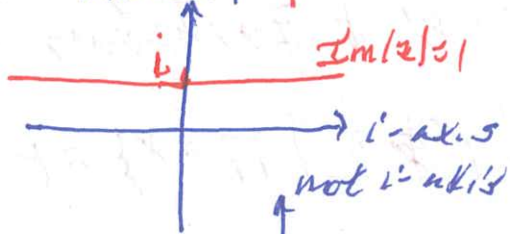
Arkan



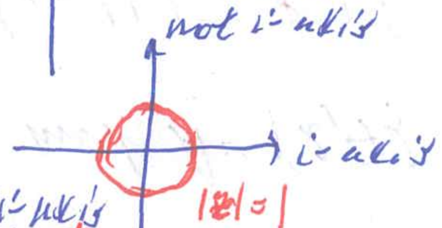
<u>Problem 1.4</u> Sketch

$\{z\in\mathbb{C} \mid Re(z)=1\}$

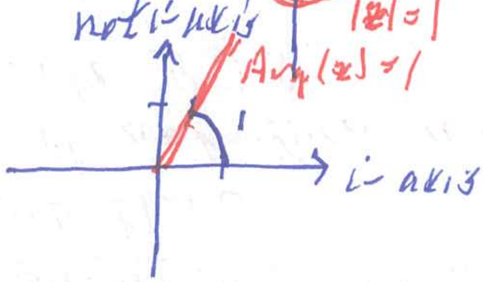
$\{z\in\mathbb{C} \mid Im(z)=1\}$



$\{z\in\mathbb{C} \mid |z|=1\}$



$\{z\in\mathbb{C} \mid Arg(z)=1\}$



Let $p(z)=z^n+c_{n-1}z^{n-1}+\dots+c_1z+c_0$

A <u>root</u> of $p(z)$ is $\alpha\in\mathbb{C}$ such that $p(\alpha)=0$

$$p(\alpha)=\alpha^n+c_{n-1}\alpha^{n-1}+\dots+c_1\alpha+c_0.$$

If $\alpha_1,\dots,\alpha_n$ are the roots of $p(z)$ then

$$p(z)=z^n+c_{n-1}z^{n-1}+\dots+c_1z+c_0=(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n).$$

<u>Example</u> $p(z) = z^4 - 1 = z^4 + 0z^3 + 0z^2 + 0z + (-1)$

$$\{\text{roots of } z^4-1\} = \{z \in \mathbb{C} \mid z^4 - 1 = 0\}$$
$$= \{z \in \mathbb{C} \mid z^4 = 1\} = \{4^{th} \text{ roots of } 1 \text{ in } \mathbb{C}\}$$
$$= \{1, i, -1, -i\} = \{e^{i\pi\frac{0}{4}}, e^{i\pi\frac{2}{4}}, e^{i\pi\frac{4}{4}}, e^{i\pi\frac{6}{4}}\}$$

and

$$z^4 - 1 = (z-1)(z-i)(z-(-1))(z-(-i)).$$

<u>Problem 2.29</u> Construct a polynomial $p(z)$ with roots $\{-2, 3+i\}$.

$$p(z) = (z-(-2))(z-(3+i)) = (z+2)(z-(3+i))$$
$$= z^2 + 2z - (3+i)z - 2(3+i)$$
$$= z^2 + 2z - 3z - iz + (-6-2i)$$
$$= z^2 + (-1-i)z + (-6-2i).$$

so

$$\{z \in \mathbb{C} \mid z^2 + (-1-i)z + (-6-2i) = 0\} = \{-2, 3+i\}.$$

<u>Fundamental theorem of complex numbers</u>

Every polynomial $p(z) = z^n + c_{n-1}z^{n-1} + \cdots + c_1 z + c_0$ with $c_0, c_1, \ldots, c_{n-1} \in \mathbb{C}$ can be factored completely, i.e., there exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$ such that

$$z^n + c_{n-1}z^{n-1} + \cdots + c_1 z + c_0 = (z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n).$$

Real coefficients

Suppose $p(z) = z^n + c_{n-1}z^{n-1} + \cdots + c_1 z + c_0$
with $c_0, c_1, \ldots, c_{n-1} \in \mathbb{R}$.

Suppose $\alpha \in \mathbb{C}$ is a root of $p(z)$. Then

$$\begin{aligned} p(\bar\alpha) &= \bar\alpha^n + c_{n-1}\bar\alpha^{n-1} + \cdots + c_1\bar\alpha + c_0 \\ &= \bar\alpha^n + \overline{c_{n-1}}\,\bar\alpha^{n-1} + \cdots + \bar c_1 \bar\alpha + \bar c_0 \quad \left(\begin{array}{l}\text{since } c_i \in \mathbb{R} \\ \text{then } \bar c_i = c_i\end{array}\right) \\ &= \overline{\alpha^n + c_{n-1}\alpha^{n-1} + \cdots + c_1\alpha + c_0} \quad \left(\begin{array}{l}\text{using that} \\ \overline{z_1 z_2} = \bar z_1 \bar z_2\end{array}\right) \\ &= \overline{p(\alpha)} = \bar 0 = 0. \end{aligned}$$

So $\bar\alpha$ is also a root of $p(z)$.

Problem 2.26 $p(z) = z^3 - 7z^2 + 17z - 15$ has real coefficients. Solve $z^3 - 7z^2 + 17z - 15 = 0$ knowing that $\alpha = 2 + i$ is a root.

First check they didn't lie:

$\alpha = 2 + i$

$\alpha^2 = 4 + 4i + i^2 = 3 + 4i$

$\alpha^3 = 6 + 11i + 4i^2 = 2 + 11i$ and

$$\begin{aligned} \alpha^3 - 7\alpha^2 + 17\alpha - 15 &= 2 + 11i - 7(3+4i) + 17(2+i) - 15 \\ &= 2 - 21 + 34 - 15 + i(11 - 28 + 17) \\ &= 0 + 0i = 0. \end{aligned}$$

So $\alpha = 2 + i$ is a root of $p(z)$

Since $p(z)$ has real coefficients and $\alpha = 2+i$ is a root of $p(z)$ then $\bar{\alpha} = 2-i$ is a root of $p(z)$.

So
$$p(z) = z^3 - 7z^2 + 17z - 15 = (z-(2+i))(z-(2-i))(z-\beta)$$
$$= (z^2 - (2+i)z - (2-i)z + (4-i^2))(z-\beta)$$
$$= (z^2 - 4z + 5)(z-\beta).$$

So $5\beta = 15$ and $\beta = 3$.

So
$$z^3 - 7z^2 + 17z - 15 = (z-(2+i))(z-(2-i))(z-3)$$
$$= (z^2 - 4z + 5)(z-3)$$

and $\{z \in \mathbb{C} \mid z^3 - 7z^2 + 17z - 15 = 0\} = \{2+i; 2-i; 3\}$.