23.03.2026 (1)
Calculus Lect. 4
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Problem 2.21

$$\left(\frac{1+i}{\sqrt{3}+i}\right)^{12} = \left(\frac{\frac{2}{\sqrt{2}}\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)}{2\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)}\right)^{12} = \left(\frac{\frac{2}{\sqrt{2}}\, e^{i\pi\frac{1}{4}}}{2\, e^{i\pi\frac{1}{6}}}\right)^{12}$$

$$= \left(2^{-1/2} e^{i\pi\frac{1}{4}} e^{-i\pi\frac{1}{6}}\right)^{12} = \left(2^{-\frac{1}{2}} e^{i\pi\left(\frac{1}{4}-\frac{1}{6}\right)}\right)^{12}$$

$$= \left(2^{-\frac{1}{2}} e^{i\pi\left(\frac{2}{24}\right)}\right)^{12} = \left(2^{-1/2} e^{i\pi\frac{1}{12}}\right)^{12} = 2^{-\frac{12}{2}} e^{i\pi\frac{12}{12}}$$

$$= 2^{-6} e^{i\pi} = \frac{1}{2^6}(-1) = \frac{-1}{64}.$$

$$\left(\frac{1-i}{\sqrt{3}-i}\right)^{7} = \left(\frac{\frac{2}{\sqrt{2}}\left(\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right)}{2\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)}\right)^{7} = \left(2^{-\frac{1}{2}}\frac{e^{-i\pi\frac{1}{4}}}{e^{-i\pi\frac{1}{6}}}\right)^{7}$$

$$= \left(2^{-1/2} e^{i\pi\left(-\frac{1}{4}+\frac{1}{6}\right)}\right)^{7} = \left(2^{-1/2} e^{i\pi\frac{-2}{24}}\right)^{7}$$

$$= \left(2^{-1/2} e^{i\pi\cdot\frac{-1}{12}}\right)^{7} = 2^{-7/2} e^{i\pi\cdot\frac{-7}{12}} = 2^{-4}\, 2^{\frac{1}{2}} e^{i\left(\frac{-7}{12}\pi\right)}$$

$$= \frac{\sqrt{2}}{16}\left(\cos\left(\frac{-7}{12}\pi\right) + i\sin\left(\frac{-7}{12}\pi\right)\right)$$

$$= \frac{\sqrt{2}}{16}\cos\left(\frac{7}{12}\pi\right) - i\sin\left(\frac{7}{12}\pi\right).$$

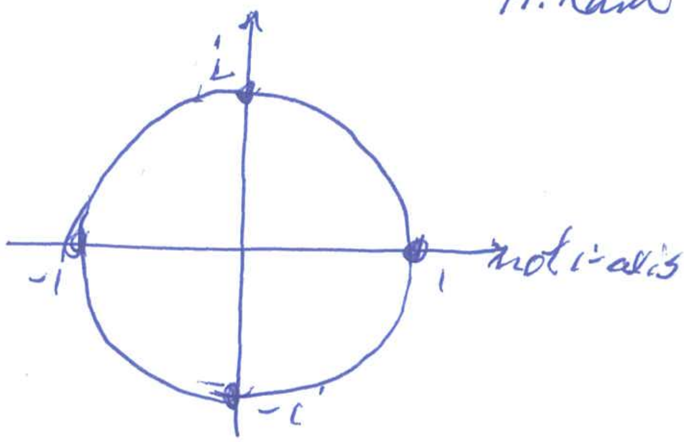
<u>Powers</u>

$1^4=1$ and $1=e^{2\pi i\cdot\frac{0}{4}}$

$i^4=1$ and $i=e^{2\pi i\cdot\frac{1}{4}}$

$(-1)^4=1$ and $-1=e^{2\pi i\cdot\frac{2}{4}}$

$(-i)^4=1$ and $-i=e^{2\pi i\cdot\frac{3}{4}}$



So $\{z\in\mathbb{C} \mid z^4-1=0\}=\{z\in\mathbb{C}\mid z^4=1\}$

$=\{1,i,-1,-i\}=\{e^{2\pi i\cdot\frac{0}{4}}, e^{2\pi i\cdot\frac{1}{4}}, e^{2\pi i\cdot\frac{2}{4}}, e^{2\pi i\cdot\frac{3}{4}}\}.$

and

$$z^4-1=(z-1)(z-i)(z-(-1))(z-(-i))$$
$$=(z-1)(z+1)(z-i)(z+i)$$
$$=(z^2-1)(z^2-i^2)=(z^2-1)(z^2+1)=z^4-1$$

<u>Problem 2.23</u> Solve $z^5=32e^{i5\pi/3}$.

$$32e^{i5\pi/3}=2^5e^{5i\pi\frac{1}{3}}$$
$$=2^5e^{5i\pi\frac{1}{3}}e^{i2\pi}$$
$$=2^5e^{5i\pi\frac{1}{3}}e^{i4\pi}$$
$$=2^5e^{5i\pi\frac{1}{3}}e^{i6\pi}$$
$$=2^5e^{5i\pi\frac{1}{3}}e^{i8\pi}$$

So

$z=2e^{i\pi\frac{1}{3}}$

or $z=2e^{i\pi\frac{1}{3}}e^{i\frac{2\pi}{5}}$

or $z=2e^{i\pi\frac{1}{3}}e^{i4\pi/5}$

or $z=2e^{i\pi\frac{1}{3}}e^{i6\pi/5}$

or $z=2e^{i\pi\frac{1}{3}}e^{i8\pi/5}$

Ex

$$\{z\in\mathbb{C} \mid z^5-32e^{i5\pi/3}=0\} = \{z\in\mathbb{C} \mid z^5 = 32e^{i5\pi/3}\}$$

$$= \left\{2e^{i\pi\frac13},\ 2e^{i\pi(\frac13+\frac25)},\ 2e^{i\pi(\frac13+\frac45)},\ 2e^{i\pi(\frac13+\frac65)},\ 2e^{i\pi(\frac13+\frac85)}\right\}$$

$$= \left\{2e^{i\pi\frac13},\ 2e^{i\pi\frac{11}{5}},\ 2e^{i\pi\frac{17}{5}},\ 2e^{i\pi\frac{23}{5}},\ 2e^{i\pi\frac{29}{5}}\right\}$$

and

$$(z-2e^{i\pi\frac13})(z-2e^{i\pi\frac{11}{5}})(z-2e^{i\pi\frac{17}{5}})(z-2e^{i\pi\frac{23}{5}})(z-2e^{i\pi\frac{29}{5}})$$

$$= z^5-32e^{i5\pi/3}.$$

$$z-1 = z-1$$

$$z^2-1 = (z-1)(z+1)$$

$$z^3-1 = (z-1)(z^2+z+1) = (z-1)(z-e^{i\pi\frac23})(z-e^{i\pi\frac43})$$

$$z^4-1 = (z-1)(z^3+z^2+z+1)$$

$$= (z-1)(z-e^{i\pi\frac24})(z-e^{i\pi\frac44})(z-e^{i\pi\frac64})$$

$$z^5-1 = (z-1)(z^4+z^3+z^2+z+1)$$

$$= (z-1)(z-e^{i\pi\frac25})(z-e^{i\pi\frac45})(z-e^{i\pi\frac65})(z-e^{i\pi\frac85})$$

$$z^6-1 = (z-1)(z^5+z^4+z^3+z^2+z+1)$$

$$= (z-1)(z-e^{i\pi\frac26})(z-e^{i\pi\frac46})(z-e^{i\pi\frac66})(z-e^{i\pi\frac86})(z-e^{i\pi\frac{10}{6}}).$$