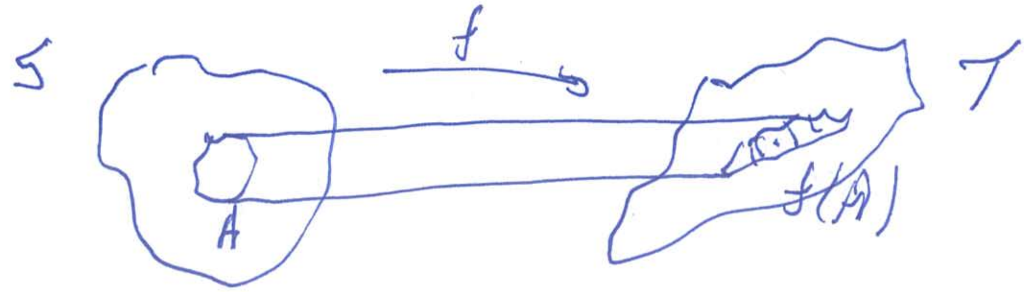
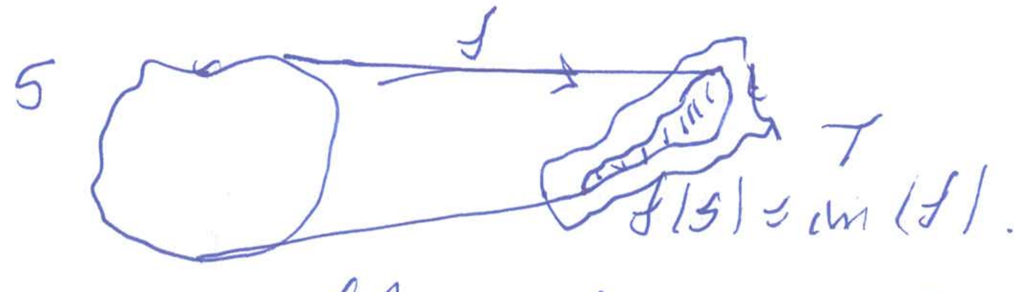
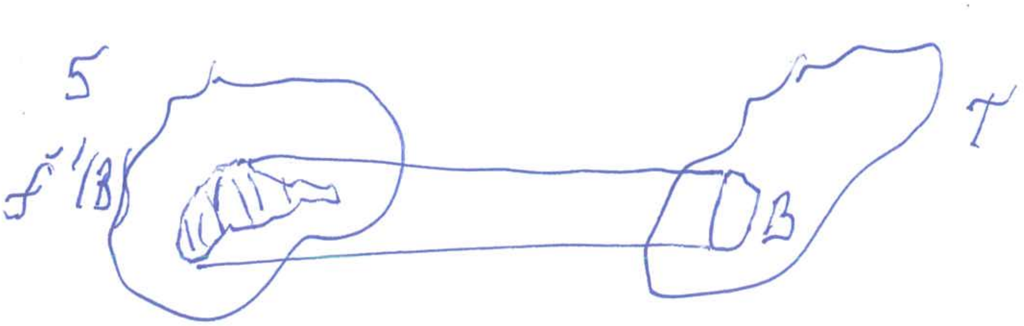
Let $f: S \to T$ be a function.
Let
$$A \subseteq S \quad \text{and} \quad B \subseteq T.$$

The <u>image of A</u> is
$$f(A) = \{ f(a) \mid a \in A \}$$
The <u>image of f</u>, or <u>range of f</u>, is
$$\mathrm{im}(f) = f(S) = \{ f(s) \mid s \in S \}$$



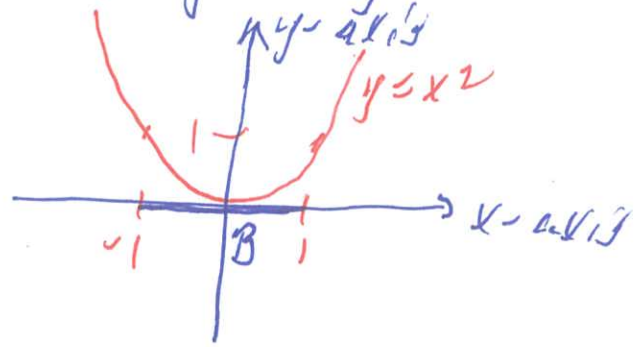


The <u>preimage of B</u>, or <u>fiber over B</u>, is
$$f^{-1}(B) = \{ a \in A \mid f(a) \in B \}$$



Problem 1.33 Let $f: \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = x^2$$



Let $A = \{-2, -1, 0, 1, 2\}$.

Then
$$f(A) = \{f(-2), f(-1), f(0), f(1), f(2)\}$$
$$= \{4, 1, 0, 1, 4\} = \{0, 1, 4\}$$

Let $B = \{x \in \mathbb{R} \mid x^2 \leq 1\} = \mathbb{R}_{[-1,1]} = [-1, 1]$

Then
$$f(B) = \{f(x) \mid -1 \leq x \leq 1\} = \{x^2 \mid -1 \leq x \leq 1\}$$
$$= \mathbb{R}_{[0,1]} = [0, 1] = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}.$$

Intervals

$\mathbb{Q}_{[a,b]} = \{x \in \mathbb{Q} \mid a \leq x \leq b\}$ $\qquad \mathbb{R}_{[a,b]} = [a, b]$

$\mathbb{Q}_{(a,b]} = \{x \in \mathbb{Q} \mid a < x \leq b\}$

$\mathbb{Q}_{[a,b)} = \{x \in \mathbb{Q} \mid a \leq x < b\}$

$\mathbb{Q}_{(a,b)} = \{x \in \mathbb{Q} \mid a < x < b\}$

$\mathbb{Q}_{[a,\infty)} = \{x \in \mathbb{Q} \mid a \leq x\}$

$\mathbb{Q}_{(a,\infty)} = \{x \in \mathbb{Q} \mid a < x\}$

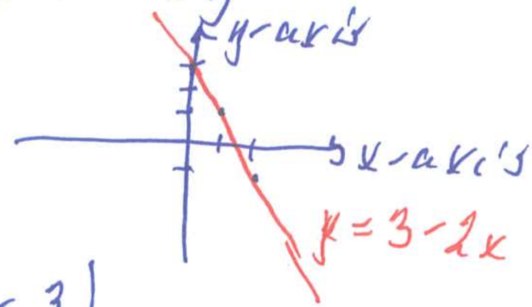
$\mathbb{Q}_{(-\infty,b)} = \{x \in \mathbb{Q} \mid x < b\}$ $\qquad (-\infty, b) = \mathbb{R}_{(-\infty,b)}$.

$\mathbb{Q}_{(-\infty,b]} = \{x \in \mathbb{Q} \mid x \leq b\}$

Problem 1.34

(1) Let $f: \mathbb{R} \to \mathbb{R}$ be given by

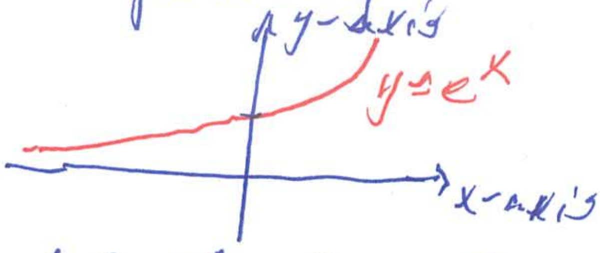
$$f(x) = 3 - 2x$$



Then $f([0,\pi]) = [3-2\pi, 3]$

(2) Let $g: \mathbb{R} \to \mathbb{R}$ be given by

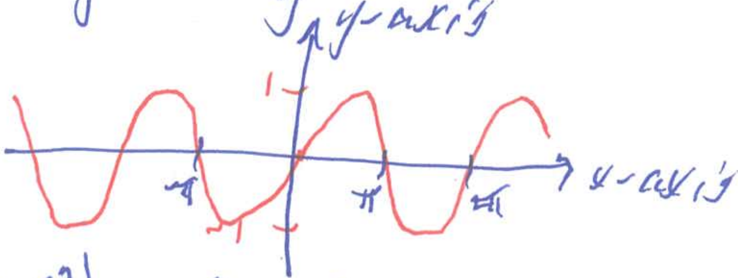
$$g(x) = e^x$$



Then $g([0,\pi]) = [e^0, e^\pi] = [1, e^\pi]$

(3) Let $h: \mathbb{R} \to \mathbb{R}$ be given by

$$h(x) = \sin(x)$$



then

$$h([0,\pi]) = \sin([0,\pi]) = [0, 1]$$

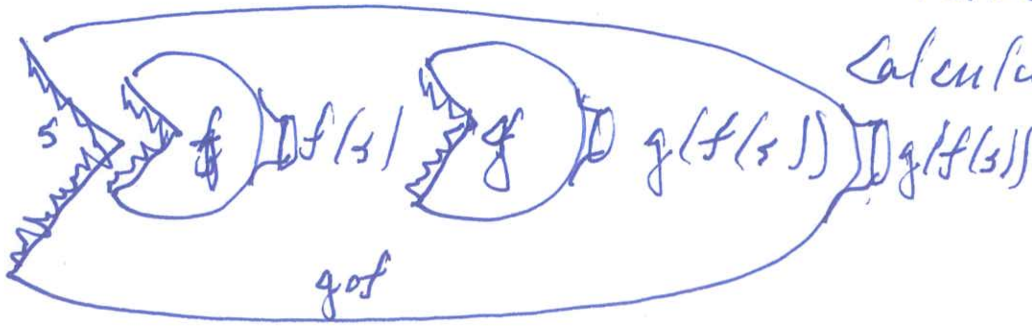
Composition and inverse functions

Let

$$f: S \to T \quad \text{and} \quad g: T \to U$$

be functions. The composition of g and f is the function

$$g \circ f: S \to U \quad \text{given by} \quad (g\circ f)(s) = g(f(s)).$$



Let S be a set. The <u>identity map on S</u> is the function $id_S : S \to S$ given by

$$id_S(s) = s. \qquad \begin{aligned} id_S : S &\to S \\ s &\mapsto s \end{aligned}$$

Let

$f: S \to T$ be a function.

The <u>inverse function to f</u> (if it exists) is the function $f^{-1}: T \to S$ such that

$$f \circ f^{-1} = id_T \quad \text{and} \quad f^{-1} \circ f = id_S.$$

In other words, if $s \in S$ and $t \in T$ then

$$f(f^{-1}(t)) = t \quad \text{and} \quad f^{-1}(f(s)) = s$$

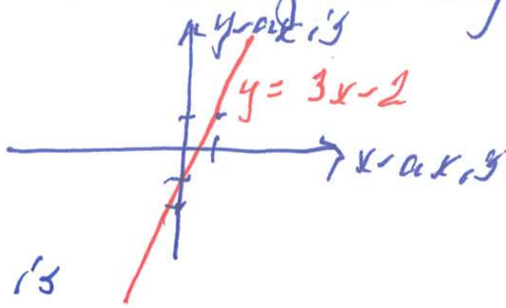
i.e. f^{-1} undoes f and f undoes f^{-1}.

Usually, f^{-1} doesn't exist.

We often try to fix f so that f^{-1} does exist.

Theorem Let $f: S \to T$ be a function. The inverse function $f^{-1}: T \to S$ exists if and only if $f: S \to T$ is bijective.

Problem 1.47 Let $f: \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 3x - 2$.



The inverse function to f is $f^{-1}: \mathbb{R} \to \mathbb{R}$ given by $f^{-1}(y) = \frac{1}{3}(y+2)$.

Check:

$$f(f^{-1}(y)) = f\left(\tfrac{1}{3}(y+2)\right) = 3\left(\tfrac{1}{3}(y+2)\right) - 2$$
$$= y + 2 - 2 = y,$$ so f undoes f^{-1}.

$$f^{-1}(f(x)) = f^{-1}(3x-2) = \tfrac{1}{3}(3x - 2 + 2) = \tfrac{1}{3}\cdot 3x = x,$$ so f^{-1} undoes f.

Favorite inverse functions that don't exist

square root: $\sqrt{x^2} = x$ and $(\sqrt{y})^2 = y$.

logarithm: $\log(e^x) = x$ and $e^{\log(y)} = y$.

arcsin: $\arcsin(\sin(x)) = x$ and $\sin(\arcsin(y)) = y$.