11.03.2026 (1)
Calculus Lect. 4
A. Ram

Functions

Functions are for comparing sets.

Let S and M be sets.

A function from S to M is an "assignment"

$$\begin{aligned} f: S &\to M \\ s &\mapsto f(s) \end{aligned}$$ which must satisfy

(a) If $s \in S$ then $f(s) \in M$

(b) If $s_1, s_2 \in S$ and $s_1 = s_2$ then $f(s_1) = f(s_2)$

The graph of f is

$$\Gamma_f = \{ (s, f(s)) \mid s \in S \}$$

A mathematically more precise definition:

A function from S to M is a subset

$\Gamma_f \subseteq S \times M$ such that

if $s \in S$ then there exists a unique $t \in M$ such that $(s, t) \in \Gamma_f$

(every student gets a single mark)

Two functions $f: S \to T$ and $g: S \to T$ are <u>equal</u> if they satisfy

if $s \in S$ then $f(s) = g(s)$. (same students same mark)

A function $f: S \to T$ is <u>injective</u> if f satisfies the condition

if $s_1, s_2 \in S$ and $f(s_1) = f(s_2)$ then $s_1 = s_2$

(every student gets a different mark or no two students get the same mark)

A function $f: S \to T$ is <u>surjective</u> if f satisfies the condition

if $t \in T$ then there exists $s \in S$ such that $f(s) = t$.

(Every possible mark is achieved by some student.)

A function $f: S \to T$ is <u>bijective</u> if f is both injective and surjective.

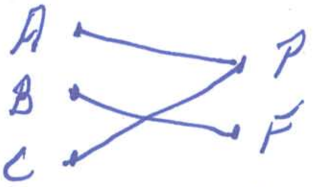
Problem 1.32 (1) $S = \{A, B, C\}$
$M = \{P, F\}$

Assignment

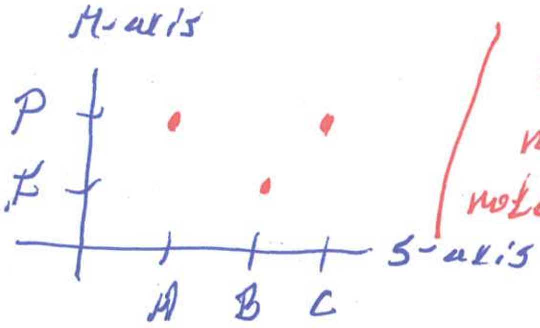
$f: S \to M$
$A \mapsto P$
$B \mapsto F$
$C \mapsto P$

Graph

$\Gamma_f = \{(A,P), (B,F), (C,P)\}$



Assignment Cartoon



Graph Cartoon

surjective, not injective, not bijective.

Problem 1.32 (2) $S = \{A, B, C\}$ and $M = \{P, F\}$.

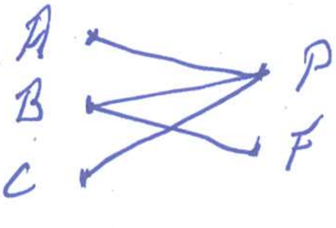
Assignment

$g: S \to M$
$A \mapsto P$
$B \mapsto P$
$B \mapsto F$
$C \mapsto P$

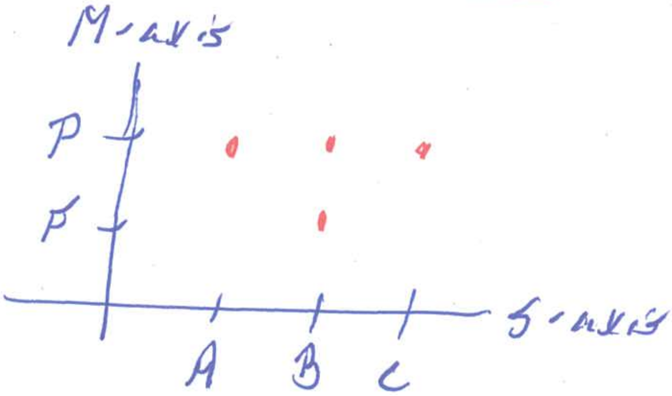
Graph

$\Gamma_g = \{(A,P), (B,P), (B,F), (C,P)\}$

NOT A FUNCTION



Assignment Cartoon

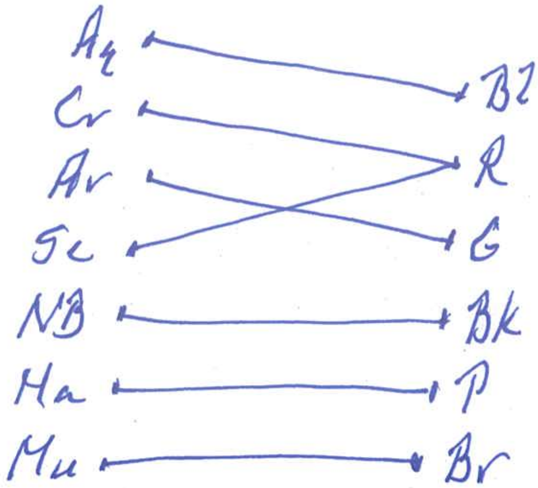


Graph Cartoon

Problem 1.41

$S = \{Aq, Cr, Ar, Sc, NB, Ma, Mu\}$

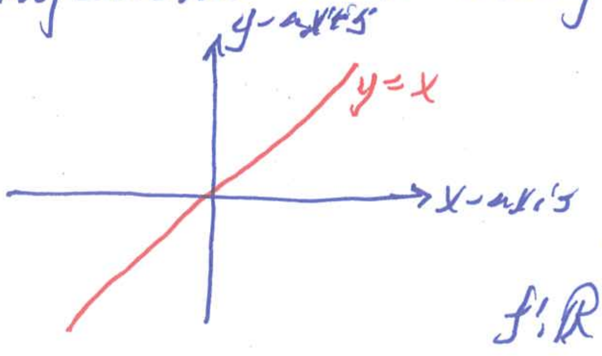
$T = \{Bl, R, G, Bk, P, Br\}$



is not injective, surjective, and not bijective.

Problem 1.39 Examples of functions $f: \mathbb{R} \to \mathbb{R}$.

(1) injective and surjective



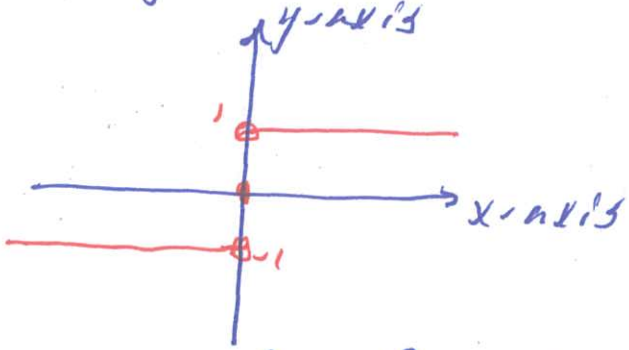
$f: \mathbb{R} \to \mathbb{R}$
$x \mapsto x.$

f is the function $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x$.

(2) bijective but not injective.

Such a function does not exist since a bijective function <u>must</u> be injective.

(3) not injective ~~but~~ and not surjective

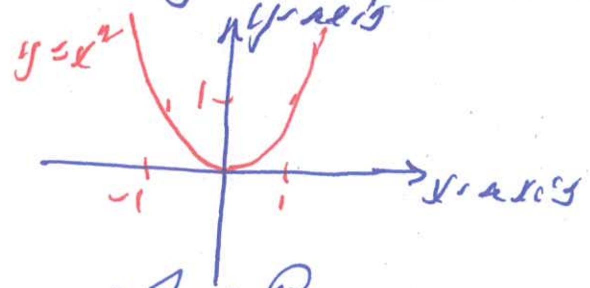


$\mathrm{sgn}: \mathbb{R} \longrightarrow \mathbb{R}$
$x \longmapsto \mathrm{sgn}(x)$

$\mathrm{sgn}: \mathbb{R} \to \mathbb{R}$ is the function given by

$$\mathrm{sgn}(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$$

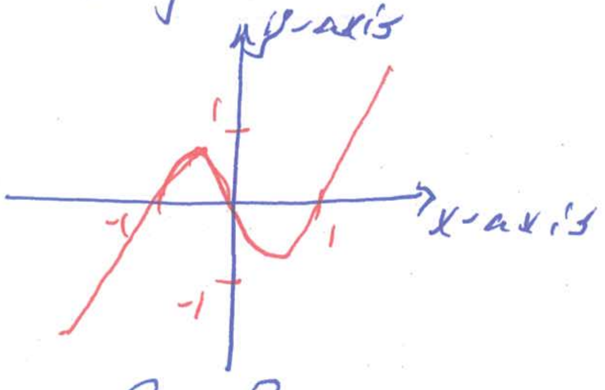
(4) not surjective ~~but~~ and not injective



The function $c: \mathbb{R} \to \mathbb{R}$ given by $c(x) = x^2$.

$c: \mathbb{R} \to \mathbb{R}$
$x \longmapsto x^2$

(5) surjective but not injective

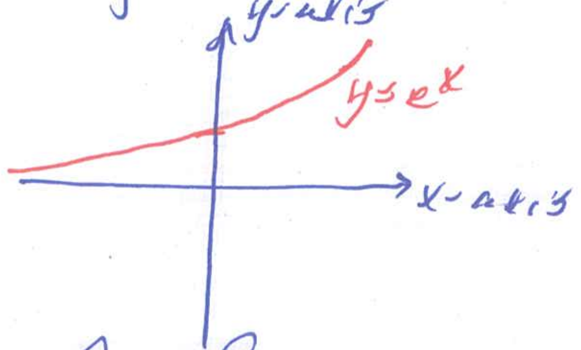


The function $d: \mathbb{R} \to \mathbb{R}$ given by

$d(x) = x(x-1)(x+1)$

$d: \mathbb{R} \to \mathbb{R}$
$x \longmapsto x^3 - x$

(6) injective and not surjective

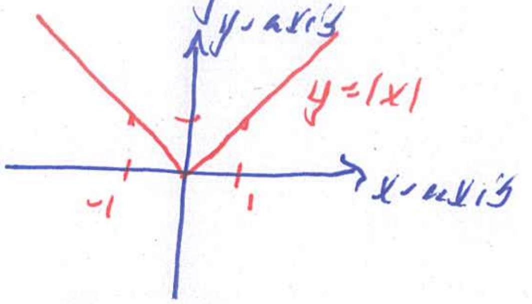


$$\begin{aligned}\mathbb{R} &\to \mathbb{R}\\ x &\mapsto e^x\end{aligned}$$

The function $\exp : \mathbb{R} \to \mathbb{R}$ given by

$$\exp(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

(7) not injective and not surjective



$$\begin{aligned}\mathbb{R} &\to \mathbb{R}\\ x &\mapsto |x|\end{aligned}$$

The function $a : \mathbb{R} \to \mathbb{R}$ given by

$$a(x) = |x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$