04.03.2026 (1)
Calculus Lect. 2
A. Ram

Numbers and intervals

The positive integers: $\mathbb{Z}_{>0} = \{1, 2, 3, \ldots\}$.

The nonnegative integers: $\mathbb{Z}_{\geq 0} = \{0, 1, 2, 3, \ldots\}$.

The integers: $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

The rational numbers:

$$\mathbb{Q} = \{ \tfrac{a}{b} \mid a \in \mathbb{Z}, b \in \mathbb{Z}_{\neq 0} \text{ and } \tfrac{a}{b} = \tfrac{c}{d} \text{ if } ad = bc \}.$$

The real numbers:

$$\mathbb{R} = \left\{ \pm a_\ell a_{\ell-1} \cdots a_1 a_0 . a_{-1} a_{-2} \cdots \;\middle|\; \begin{matrix} \ell \in \mathbb{Z}_{\geq 0} \\ a_i \in \{0, 1, \ldots 9\} \\ a_\ell \neq 0 \text{ if } \ell > 0 \end{matrix} \right\}$$

with a requirement that if $a_k \neq 9$ then

$$\pm a_\ell \cdots a_{k+1} a_k 999 \ldots = \pm a_\ell \cdots a_{k+1} (a_k + 1) 000 \ldots$$

(so that, for example, $0.999\ldots = 1.000\ldots$).

The complex numbers:

$$\mathbb{C} = \{ a + bi \mid a, b \in \mathbb{R} \} \text{ with } i^2 = -1.$$

Then

$$\mathbb{Z}_{>0} \subseteq \mathbb{Z}_{\geq 0} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}.$$

The real numbers have an order. Calculus Lect. 2 A. Ram

(The complex numbers do not have an order.)

<u>Axioms for the order</u> (Axioms for inequalities)

(1) If $a, b, c \in \mathbb{R}$ and $a < b$ then $a + c < b + c$.

(2) If $a, b, c \in \mathbb{R}$ and

$$a < b \text{ and } c > 0 \text{ then } ac < bc.$$

(3) If $a, b, c \in \mathbb{R}$ and

$$a < b \text{ and } c < 0 \text{ then } ac > bc.$$

<u>Problem 0.2</u> Solve the inequality

$$\frac{x+4}{x-2} < 1.$$

Use a similar method to solving $\frac{x+4}{x-2} = 1$.

Multiply both sides by $x-2$ to get

$$x + 4 < x - 2 \quad \text{if } x - 2 > 0,$$

$$x + 4 > x - 2 \quad \text{if } x - 2 < 0$$

Subtract $x-2$ from both sides to get.

$$6 < 0 \quad \text{when } x > 2$$

$$6 > 0 \quad \text{when } x < 2.$$

Since 6 is never less than 0 then

$$x < 2.$$

04.03.2026 ③
Calculus Lect. 2
A. Ram

Problem 1.25 = Exercise 1.1.10

Solve $-3x+5 \le 2$.

Add negative 5 to both sides to get
$$-3x \le -3.$$
Multiply both sides by $\frac{-1}{3}$ to get
$$x \geq 1.$$

Induction (The definition of $\mathbb{Z}_{>0}$).

Problem 1.28 Prove, by induction, that

if $n \in \mathbb{Z}_{>0}$ then $1+2+\cdots+n = \frac{n(n+1)}{2}$.

Proof Base case: To show: $1 = \frac{1(1+1)}{2}$.

$RHS = \frac{1\cdot(1+1)}{2} = 1 \cdot \frac{2}{2} = 1 \cdot 1 = 1 = LHS.$

Induction step: Assume $1+2+\cdots+n = \frac{n(n+1)}{2}$.

To show: $1+2+\cdots+n+(n+1) = \frac{(n+1)(n+1+1)}{2}$

$$\begin{aligned} LHS &= 1+2+\cdots+n+(n+1) \\ &= \frac{n(n+1)}{2} + (n+1) \\ &= (n+1)\left(\frac{n}{2}+1\right) = (n+1)\left(\frac{n+2}{2}\right) = \frac{(n+1)(n+1+1)}{2} \\ &= R.H.S. \quad // . \end{aligned}$$

Problem 1.25(b) Prove, by induction, that if $n \in \mathbb{Z}_{>0}$ then $2^n \geq n+1$.

Proof Base case: To show: $2^1 \geq 1+1$.

$LHS = 2^1 = 2 = 1+1 \geq 1+1 = RHS$.

Induction step: Assume $2^n \geq n+1$.

To show: $2^{n+1} \geq n+1+1$

$LHS = 2^{n+1} = 2^n \cdot 2 \geq (n+1)\cdot 2 = n+1+n+1$

$\geq n+1+1 = RHS$. //

Problem 1.25(a) Prove, by induction, that if $n \in \mathbb{Z}_{>0}$ then $1+3+5+\cdots+(2n-1) = n^2$.

Proof: Base case: $1 = 1^2$.

$RHS = 1^2 = 1\cdot 1 = 1 = LHS$

Induction step: Assume $1+3+\cdots+(2n-1) = n^2$.

To show: $1+3+\cdots+(2n-1)+(2(n+1)-1) = (n+1)^2$.

$LHS = 1+3+\cdots+(2n-1)+(2(n+1)-1)$

$= n^2+2(n+1)-1 = n^2+2n+2-1$

$= n^2+2n+1$.

$RHS = (n+1)^2 = (n+1)(n+1) = n^2+2n+1$.

So $LHS = RHS$. //

Gauss in Kindergarten 04.03.2026 (5)

Calculus Lect. 2

A. Ram.

Prove, without using induction, that

if $n \in \mathbb{Z}_{>0}$ then $1+2+\cdots+n = \frac{n(n+1)}{2}$

$$
\begin{aligned}
2LHS = \quad & LHS = 1+2+\cdots+(n-1)+n \\
+ & LHS \quad\;\; n+(n-1)+\cdots+2+1 \\
& = (n+1)+(n+1)+\cdots+(n+1)+(n+1) \\
& = n(n+1).
\end{aligned}
$$

So $LHS = \frac{n(n+1)}{2}$. //.

For example:

$$
\begin{aligned}
& 1+2+3+\cdots+98+99+100 \\
+ & 100+99+98+\cdots+3+2+1 \\
= & 101+101+101+\cdots+101+101+101 \\
= & 10100
\end{aligned}
$$

So $1+2+\cdots+100 = \frac{10100}{2} = 5050$.