

The Koszul complex or de Rham complex

$$0 \rightarrow \mathcal{O}_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \Omega^2_X \xrightarrow{d} \dots \xrightarrow{d} \Omega^{\text{top}}_X \rightarrow 0$$

is a complex of \mathcal{O}_X -modules.

\mathcal{O}_X = ring of functions on X

T_X = tangent bundle

$\Omega^1_X = T_X^* =$ cotangent bundle
= module of differentials

$\Omega^p_X = \wedge^p \Omega^1_X =$ module of p -forms

$\Omega^{\text{top}}_X =$ canonical bundle.

If $X = \text{Spec}(\mathbb{C}[x_1, \dots, x_n])$, the sections of

T_X have \mathcal{O}_X -basis $\left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right\}$

Ω^1_X have \mathcal{O}_X -basis $\{dx_1, \dots, dx_n\}$

Ω^p_X have \mathcal{O}_X -basis $\left\{ dx_{i_1} \wedge \dots \wedge dx_{i_p} \mid \begin{array}{l} i_1, \dots, i_p \in \{1, \dots, n\} \\ i_1 < \dots < i_p \end{array} \right\}$

\mathcal{O}_X has \mathcal{O}_X -basis $\{1\}$

(see Harder §4.3.3 p. 63 and §4.10.2 p. 156)

(see also §7.5.3 Examples 15, 16)

The Weyl algebra Let $[a, b] = ab - ba$.

Let $X = \text{Spec}(\mathbb{C}[x_1, \dots, x_n])$.

The symmetric algebra is

$\mathcal{O}_X = \mathbb{C}[x_1, \dots, x_n]$, the \mathbb{C} -algebra generated by x_1, \dots, x_n with relations

$$[x_i, x_j] = 0 \quad \text{for } i, j \in \{1, \dots, n\}$$

The Weyl algebra, or Heisenberg algebra, is

$$D_X = \mathbb{C}[x_1, \dots, x_n] \otimes \mathbb{C}\left[\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right],$$

the \mathbb{C} -algebra generated by $x_1, \dots, x_n, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ with relations

$$[x_i, x_j] = 0, \quad \left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0, \quad \left[\frac{\partial}{\partial x_j}, x_i\right] = \delta_{ij}.$$

The exterior algebra is

$$\Omega_X = \Lambda(\Omega'_X),$$

the \mathbb{C} -algebra with product \wedge , generated by dx_1, \dots, dx_n with relations

$$dx_i \wedge dx_j = -dx_j \wedge dx_i, \\ \text{for } i, j \in \{1, \dots, n\}.$$

Constructing the tangent bundle T_X N. Lane

Let \mathcal{O}_X be a ~~ring~~ \mathbb{C} -algebra

A derivation is a \mathbb{C} -linear map $\partial: \mathcal{O}_X \rightarrow \mathcal{O}_X$ such that

$$\partial(f_1 f_2) = f_1 (\partial f_2) + (\partial f_1) f_2.$$

Let (X, \mathcal{O}_X) be a ringed space.

Let $p \in X$. A tangent vector to X at p is a \mathbb{C} -linear map $\eta: \mathcal{O}_X \rightarrow \mathbb{C}$ such that

$$\eta(f_1 f_2) = f_1(p) \eta(f_2) + \eta(f_1) f_2(p)$$

for $f_1, f_2 \in \mathcal{O}_X$. The tangent bundle to X is

$$T_X = \text{Hom}_{\mathbb{C}\text{-alg}} \left(\mathcal{O}_X, \frac{\mathbb{C}[t]}{t^2 \mathbb{C}[t]} \right) \text{ with } \begin{array}{c} T_X \\ \downarrow t=0 \\ X \end{array}$$

If $\gamma \in T_X$ and $\zeta: \mathcal{O}_X \rightarrow \mathbb{C}$ and $\eta: \mathcal{O}_X \rightarrow \mathbb{C}$ are such that

$$\gamma = \zeta + t\eta$$

then

$$\zeta(f_1 f_2) = \zeta(f_1) \zeta(f_2)$$

(see Harder equation (7.19))

and

$$\eta(f_1 f_2) = \zeta(f_1) \eta(f_2) + \eta(f_1) \zeta(f_2)$$

Normally one identifies

$$\left\{ \text{points of } X \right\} \longleftrightarrow \text{Hom}_{\mathbb{C}\text{-alg}}(\mathcal{O}_X, \mathbb{C})$$

$$p \longmapsto \text{ev}_p$$

where $\text{ev}_p(f) = f(p)$. With this notation

$$\delta = \text{ev}_p + t\eta \quad (\text{and } \xi = \text{ev}_p)$$

and

$$\eta(f, h) = f_1(p)\eta(h_1) + \eta(h_1, f_2(p)).$$

A vector field? is a section ∂ of T_X ,
i.e. a choice of tangent vector at each
point $p \in X$. So

$\partial: \mathcal{O}_X \rightarrow \mathcal{O}_X$ is a \mathbb{C} -linear map
such that

$$\partial(f, h) = f_1(\partial h) + (\partial f_1)h_2.$$

Hence

$$\left\{ \begin{array}{l} \text{derivations} \\ \text{of } \mathcal{O}_X \end{array} \right\} = \left\{ \begin{array}{l} \text{vector fields} \\ \text{on } X \end{array} \right\} = \left\{ \begin{array}{l} \text{sections} \\ \text{of } T_X \end{array} \right\}$$

Constructing the cotangent bundle Ω_X A. Lew (5)

The module of differentials, is the pair (Ω_X', d) where Ω_X' is an \mathcal{O}_X -module and $d: \mathcal{O}_X' \rightarrow \Omega_X'$ is a \mathbb{C} -linear map such that

$$d(f_1 f_2) = f_1 d(f_2) + f_2 d(f_1)$$

which satisfies the following universal property:

If \mathcal{M} is a \mathcal{O}_X -module and $d_1: \mathcal{O}_X \rightarrow \mathcal{M}$ is a \mathbb{C} -linear map such that

$$d_1(f_1 f_2) = f_1 d_1(f_2) + f_2 d_1(f_1)$$

then there exists a unique \mathcal{O}_X -module morphism $\varphi: \Omega_X' \rightarrow \mathcal{M}$ with $\varphi \circ d = d_1$

$$\begin{array}{ccc} \mathcal{O}_X & \xrightarrow{d_1} & \mathcal{M} \\ d \searrow & & \nearrow \varphi \\ & \Omega_X' & \end{array}$$

(see Harder
Prop. 7.5.10(a))

Existence of (Ω_X', d) : The diagonal map is

$$\begin{array}{ccc} X & \xrightarrow{\Delta} & X \times X \\ p_1 & \mapsto & (p, p) \end{array}$$

and

$$\begin{array}{ccc} \mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{O}_X & \xrightarrow{\text{Spec}(\Delta)} & \mathcal{O}_X \\ f_1 \otimes f_2 & \mapsto & f_1 f_2 \end{array}$$

Then $\text{Spec}(\Delta)$ is an \mathcal{O}_X -module map where

$$f(g_1 \otimes g_2) = f g_1 \otimes g_2 \text{ specifies the } \mathcal{O}_X \text{-action on } \mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{O}_X.$$

The cotangent bundle Ω'_X is (Harder §7.5.2)

$$\Omega'_X = \frac{\mathcal{I}}{\mathcal{I}^2} \quad \text{where } \mathcal{I} = \ker \left(\begin{array}{c} \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X \xrightarrow{\text{Spec}(\Delta)} \mathcal{O}_X \\ f_1 \otimes f_2 \mapsto f_1 f_2 \end{array} \right)$$

with the \mathcal{O}_X -module map

$$d: \mathcal{O}_X \rightarrow \Omega'_X \quad \text{given by } df = f \otimes 1 - 1 \otimes f.$$

Since \mathcal{I} is an \mathcal{O}_X -module then Ω'_X is an \mathcal{O}_X -module. The map $d: \mathcal{O}_X \rightarrow \Omega'_X$ is surjective (see Harder, Lemma 7.5.7).

HW: Show that the map

$$\text{Hom}_{\mathcal{O}_X}(\Omega'_X, \mathcal{O}_X) \longrightarrow \left\{ \begin{array}{l} \text{derivations} \\ \text{of } \mathcal{O}_X \end{array} \right\}$$

$$w \longmapsto w \circ d$$

is an isomorphism of \mathcal{O}_X -modules, so that the tangent bundle T_X is the dual of Ω'_X . (see Harder §7.5.5).

HW: (The chain rule).

A morphism $\varphi: X \rightarrow Y$ corresponds to a morphism

$$\varphi^*: \mathcal{O}_Y \rightarrow \mathcal{O}_X \quad \text{given by } \varphi^*(f) = f \circ \varphi. \quad \text{Then}$$

$$\text{there is } T_\varphi: T_X \rightarrow T_Y \quad \text{and } T_{\varphi \circ \psi} = T_\varphi \circ T_\psi.$$

so that T is a "functor".

The Koszul complex

The exterior algebra

$\Omega_X = \Lambda(\Omega'_X)$ is an \mathcal{O}_X -algebra with \wedge as a product. Then Ω_X is a graded algebra $\Omega_X = \bigoplus_{p \in \mathbb{Z}, 0} \Omega_X^p$ with

$$\Lambda^0(\Omega'_X) = \mathcal{O}_X, \quad \Lambda^1(\Omega'_X) = \Omega'_X \quad \text{and} \quad \Omega_X^p = \Lambda^p(\Omega'_X).$$

The Koszul complex

$$0 \rightarrow \mathcal{O}_X \xrightarrow{d} \Omega'_X \xrightarrow{d} \Omega_X^2 \xrightarrow{d} \dots \xrightarrow{d} \Omega_X^{\text{top}} \rightarrow 0$$

is obtained by extending $\mathcal{O}_X \xrightarrow{d} \Omega'_X$ by

$$d^2 = 0 \quad \text{and}$$

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^{\deg(\omega_1) \deg(d)} \omega_1 \wedge d\omega_2$$

where $\deg(d) = 1$. (Harder §4.10.1 p 151 and Bourbaki Alg. Ch. X, Algèbre Homologique Prop 13).

HW Look up the definition of differential graded algebra (dg algebra) and show that (Ω_X, d) is a dg algebra.

Let \mathcal{M} be a vector bundle on X (i.e. a locally free \mathcal{O}_X -module).

A connection on \mathcal{M} is a \mathcal{O} -module morphism

$$\nabla: \mathcal{M} \rightarrow \mathcal{M} \otimes_{\mathcal{O}_X} \Omega_X^1$$

such that

$$\nabla(mf) = \nabla(m)f + mdf$$

for $f \in \mathcal{O}_X$ and $m \in \mathcal{M}$. Define

$$0 \rightarrow \mathcal{M} \xrightarrow{\nabla} \mathcal{M} \otimes \Omega_X^1 \xrightarrow{\nabla} \mathcal{M} \otimes \Omega_X^2 \xrightarrow{d} \dots \xrightarrow{d} \mathcal{M} \otimes \Omega_X^{\text{top}} \rightarrow 0$$

by

$$\nabla(m \otimes w) = m \otimes dw + \nabla(m) \wedge w$$

for $m \in \mathcal{M}$ and $w \in \Omega_X^p$.

The connection ∇ is flat if

$$\nabla^2 = 0.$$

Let M be a vector bundle on X (i.e. a locally free \mathcal{O}_X -module).

Choose a (local) basis of M

e_1, e_2, \dots, e_n so that $M = \mathcal{O}_X^{\oplus n}$

is (locally) column vectors of length n with entries in \mathcal{O}_X ; if $m \in M$ then

$$m = e_1 f_1 + \dots + e_n f_n = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \text{ with respect to the basis } e_1, \dots, e_n.$$

Let $\nabla: M \rightarrow M \otimes_{\mathcal{O}_X} \Omega_X$ be a connection and define $A_{ij} \in \Omega_X$ by

$$\nabla(e_i) = e_1 \otimes A_{1i} + \dots + e_n \otimes A_{ni}.$$

Then

$$\begin{aligned} \nabla(m) &= \nabla(f_1 e_1) + \dots + \nabla(f_n e_n) \\ &= \nabla(e_1) f_1 + e_1 df_1 + \dots + \nabla(e_n) f_n + e_n df_n \\ &= e_1 \otimes A_{11} f_1 + \dots + e_n \otimes A_{n1} f_1 + e_1 df_1 \\ &\quad + e_2 \otimes A_{12} f_2 + \dots + e_n \otimes A_{n2} f_2 + e_2 df_2 \\ &\quad + \dots + \\ &\quad + e_n \otimes A_{nn} f_n + \dots + e_n \otimes A_{nn} f_n + e_n df_n \end{aligned}$$

$$= e_1 (df_1 + A_{11}s_1 + A_{12}s_2 + \dots + A_{1n}s_n) + e_2 (df_2 + A_{21}s_1 + A_{22}s_2 + \dots + A_{2n}s_n) + \dots + e_n (df_n + A_{n1}s_1 + A_{n2}s_2 + \dots + A_{nn}s_n)$$

$$= \begin{pmatrix} df_1 \\ \vdots \\ df_n \end{pmatrix} + \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{pmatrix} \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix}$$

$$= (d + A) \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix} \quad \text{where } A = \begin{pmatrix} A_{11} & \dots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \dots & A_{nn} \end{pmatrix}$$

is an $n \times n$ matrix with entries in \mathcal{O}_x

and $d \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix} = \begin{pmatrix} df_1 \\ \vdots \\ df_n \end{pmatrix}$

Let $w_i = df_i + s_1 A_{i1} + s_2 A_{i2} + \dots + s_n A_{in}$

so that

$$\nabla(m) = e_1 \otimes w_1 + \dots + e_n \otimes w_n.$$

Then

$$\nabla^2(m) = \nabla(\nabla(m)) = \nabla(e_1 \otimes w_1 + \dots + e_n \otimes w_n)$$

$$= \nabla(e_1) \wedge w_1 + e_1 \otimes dw_1 + \dots + \nabla(e_n) \wedge w_n + e_n \otimes dw_n$$

$$\begin{aligned}
 &= e_1 \otimes A_{11} \wedge \omega_1 + \dots + e_n \otimes A_{n1} \wedge \omega_1 + e_1 \otimes d\omega_1 \quad A. \text{Law} \\
 &+ e_1 \otimes A_{12} \wedge \omega_2 + \dots + e_n \otimes A_{n2} \wedge \omega_2 + e_1 \otimes d\omega_2 \\
 &+ \dots \\
 &+ e_1 \otimes A_{1n} \wedge \omega_n + \dots + e_n \otimes A_{nn} \wedge \omega_n + e_n \otimes d\omega_n \\
 &= e_1 \otimes (A_{11} \wedge \omega_1 + A_{12} \wedge \omega_2 + \dots + A_{1n} \wedge \omega_n + d\omega_1) \\
 &+ e_2 \otimes (A_{21} \wedge \omega_1 + A_{22} \wedge \omega_2 + \dots + A_{2n} \wedge \omega_n + d\omega_2) \\
 &+ \dots \\
 &+ e_n \otimes (A_{n1} \wedge \omega_1 + A_{n2} \wedge \omega_2 + \dots + A_{nn} \wedge \omega_n + d\omega_n).
 \end{aligned}$$

Since $\omega_i = df_i + f_1 A_{i1} + \dots + f_n A_{in}$ then

$$\begin{aligned}
 d\omega_i &= d^2 f_i + df_1 \wedge A_{i1} + f_1 dA_{i1} + \dots + df_n \wedge A_{in} + f_n dA_{in} \\
 &= 0 + \sum_{j=1}^n df_j \wedge A_{ij} + \sum_{k=1}^n dA_{ik} f_k
 \end{aligned}$$

and

$$\begin{aligned}
 \sum_{j=1}^n A_{ij} \wedge \omega_j &= \sum_{j=1}^n A_{ij} \wedge (df_j + f_1 A_{j1} + \dots + f_n A_{jn}) \\
 &= \left(- \sum_{j=1}^n df_j \wedge A_{ij} \right) + \sum_{j,k=1}^n A_{ij} \wedge A_{jk} f_k
 \end{aligned}$$

So

$$\nabla^2(m) = \sum_{i=1}^n e_i \otimes (A_{i1} \wedge \omega_1 + \dots + A_{in} \wedge \omega_n + d\omega_i)$$

$$= \sum_{i=1}^n e_i \otimes \left(- \sum_{j=1}^n d f_j \wedge A_{ij} + \sum_{j,k=1}^n A_{ij} \wedge A_{jk} f_k \right. \\ \left. + \sum_{j=1}^n d f_j \wedge A_{ij} + \sum_{k=1}^n d A_{ik} f_k \right)$$

$$= \sum_{i=1}^n e_i \otimes \left(\sum_{k=1}^n (d A_{ik} + \sum_{j=1}^n A_{ij} \wedge A_{jk}) f_k \right)$$

Letting

$$dA = (dA_{ik})_{1 \leq i, k \leq n} \text{ and } A \wedge A = \left(\sum_{j=1}^n A_{ij} \wedge A_{jk} \right)_{1 \leq i, k \leq n}$$

which are $n \times n$ matrices with entries on Ω_X^2
then

$$\nabla^2 \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = (dA + A \wedge A) \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \text{ in the basis } e_1, \dots, e_n \text{ of } \mathcal{M}.$$

The curvature of ∇ is the matrix

$$F = dA + A \wedge A, \text{ where}$$

$A = (A_{ij})$ is given by

$$\nabla(e_i) = e_i \otimes A_{i1} + \dots + e_n \otimes A_{ni}.$$

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We have proved,

V is flat if and only if

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$F = 0$.

(13)

(a) From Quentin Bell:

"An important skill, especially in research, is asking questions. Your assignments are a form of research. As such they should have many questions (Why this? How is this useful? Is this similar to that?) While assignments should be drafted and revised before submission, including these questions will allow you to get good feedback on this aspect of research." He has said it perfectly.

(b) There are multiple (two) standardly used topologies on \mathbb{P}^n used in algebraic geometry (denoted \mathbb{P}^n and $\hat{\mathbb{P}}^n$ in the Week 2 lecture notes). An important part of the understanding of \mathbb{P}^n in algebraic geometry (and for the assignment, is to compare and contrast these topologies and determine which properties of \mathbb{P}^n hold in each case.

(c) The definition (and topology) of S^2 is in the lecture notes Week 2 (additional notes and main notes). If you wish to use a different topology on S^2 that is fine, but give a careful definition, explain why you choose this alternate topology, how it relates to the topology used in class and how its relation to the two topologies on \mathbb{P}^1 is affected by this alternate choice.

(d) As evidenced by the heading "Projective space as a ringed space" in the lecture notes for Week 2, it is vital, in algebraic geometry not only to view projective space as a topological space, but, in many contexts, as a topological space with additional structure (an object in the category of ringed spaces). It is part of the subject to learn in which contexts this extra structure

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is an important part of the statements being made and in which contexts it is not.

(e) Proper referencing in mathematical writing is important to ensure that the writing contains no plagiarism (even unintentionally). Referencing, and what the goals and the culture are was discussed at some length in class. In many cases, during the marking of assignments, feedback is given on how to be more attentive, more accurate and more informative with referencing techniques and styles.

(f) Affine varieties space is defined in Harder, Definition 6.2.10.

(g) An important component of research training is the development of skills for using resources to obtain information. The definition of perfectoid space from Scholze's answer on MathOverflow

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or from Scholze's PhD thesis, provided
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it is given with a proper reference to the source, constituted a good answer to one part of one question or assignment 1.

This definition is "A perfectoid space is a ringed space which is locally isomorphic to an affinoid perfectoid space". This definition has a very strong similarity to definitions of varieties, manifolds and schemes as given in class and in the lecture notes. This makes the definition of perfectoid space fit squarely into the topic and material covered in lectures. As explained in lecture and in the lecture notes, it is the term "locally isomorphic" which has a particular need for further amplification in the context of the material for this subject.

(h) There are headings "Affine space as a ringed space" and "Projective space as a ringed space" which strongly indicate that viewing these objects in the category of sets is not on the spirit of this course.

- (i) An important goal of the course is to teach, learn and improve the quality of one's mathematical writing and mathematical exposition. The final marks for assignments (and exams) are reflective of an assessment of the exposition, the clarity, the correctness and the depth of the exploration. This mechanism for assessment is similar to assessment of an essay, a report, or a mathematics paper. The assignments do provide additional structure, as a bridge between lower level ^{mathematics} assignments and mathematical papers, but are intended to begin the process of learning a more advanced approach to mathematical exposition and thoughtful analysis.