08.04.2026

Xiamen talk (1)

A. Ram

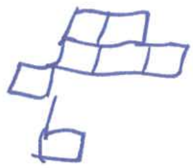
Macdonald polynomials

based on jt work with L. Colmenarejo.

Polynomials

$\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ has basis $\{x^\mu \mid \mu \in \mathbb{Z}^n\}$

$x^\mu = x_1^{\mu_1} \cdots x_n^{\mu_n}$ for $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$

Example $\mu = (2, 3, -1, 0, 1) =$ 

$x^\mu = x_1^2 x_2^3 x_3^{-1} x_5$.

Alternating symmetric functions

S_n acts on $\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ by permuting $x_1, \dots, x_n$.
S_n acts on $\mathbb{Z}^n$ by permuting coordinates.

$$a_{\lambda+\rho} = \sum_{w \in S_n} \operatorname{sgn}(w)\, w x^{\lambda+\rho} = \sum_{w \in S_n} \det(w)\, x^{w(\lambda+\rho)}$$

for $\lambda = (\lambda_1, \dots, \lambda_n)$ and $\rho = (n-1, \dots, 2, 1, 0)$

Monomial symmetric functions

$$m_\lambda = \sum_{\gamma \in S_n \lambda} x^\gamma = \frac{1}{K_\lambda} \sum_{w \in S_n} w x^\lambda.$$

Classical world		qt world	
x^μ	specialize	$E_\mu = E_\mu(q,t)$	electronic
$a_{\lambda+\rho}$	$\leftsquigarrow$ q and t	$A_{\lambda+\rho} = A_{\lambda+\rho}(q,t)$	fermionic
m_λ		$P_\lambda = P_\lambda(q,t)$	bosonic

08.09.2024

The E_μ: electronic, or nonsymmetric, Xiamen Talk 2
Macdonald polynomials. A. Ram

Fix $q, t^{\frac{1}{2}} \in \mathbb{C}^\times$. Define operators on $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$

$$T_i f = t^{\frac{1}{2}} f + (1+s_i) \frac{t^{-\frac{1}{2}} - t^{\frac{1}{2}} x_i^{-1} x_{i+1}}{1 - x_i^{-1} x_{i+1}} f$$

where $s_i = (i, i+1)$ is the transposition in S_n switching i and $i+1$.

$$(T_\pi f)(x_1, \ldots, x_n) = f(q^{-1} x_n, x_1, x_2, \ldots, x_{n-1}).$$

$$Y_1 = T_\pi T_{n-1} \cdots T_2 T_1 \quad \text{and} \quad Y_{j+1} = T_j^{-1} Y_j T_j^{-1}.$$

There is a unique basis

$$\{E_\mu \mid \mu \in \mathbb{Z}^n\} \quad \text{of} \quad \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$$

determined by

$$Y_i E_\mu = q^{-\mu_i} t^{-(v_\mu(i) - 1) + \frac{1}{2}(n-1)} E_\mu$$

and the coefficient of x^μ in E_μ equals 1.

v_μ is the permutation in S_n that rearranges μ in (weakly) increasing order.

The $A_{\lambda+\rho}$: fermionic Macdonald polynomials.

For $w \in S_n$ define

$$T_w = T_{i_1} T_{i_2} \cdots T_{i_\ell} \quad \text{if} \quad w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$$

with ℓ minimal ($\ell = \ell(w)$ is the length of w).

$$A_{\lambda+\rho} = \left(\prod_{1 \le i < j \le n} \frac{x_i - t x_j}{x_i - x_j} \right) \sum_{w \in S_n} \mathrm{sgn}(w)\, w E_{\lambda+\rho}$$

$$= t^{\frac{1}{2}\binom{n}{2}} \sum_{w \in S_n} (-t^{-\frac{1}{2}})^{\ell(w)} T_w E_{\lambda+\rho}$$

$$= \sum_{\mu \in S_n(\lambda+\rho)} (\mathrm{const}_\mu) E_\mu, \quad \text{where}$$

$$\mathrm{const}_\mu = \prod_{1 \le i < j \le n} \left((-1) \left(\frac{1 - q^{\mu_i - \mu_j} t^{v_\mu(i) - v_\mu(j) + 1}}{1 - q^{\mu_i - \mu_j} t^{v_\mu(i) - v_\mu(j)}} \right) \right)$$

The P_λ: bosonic, or symmetric, Macdonald polynomials.

$$P_\lambda = \frac{1}{K_\lambda(t)} \sum_{w\in S_n} w\left(E_\lambda \prod_{1\le i<j\le n} \frac{x_i - t x_j}{x_i - x_j}\right)$$

$$= \frac{1}{K_\lambda(t)} t^{\frac{1}{2}\binom{n}{2}} \sum_{w\in S_n} T_w E_\lambda$$

$$= \sum_{\gamma\in S_n\lambda} (\mathrm{CONST}_\gamma)\, E_\gamma, \quad \text{where}$$

$$\mathrm{CONST}_\gamma = \prod_{\substack{1\le i<j\le n\\ \gamma_i>\gamma_j}} \left(t\left(\frac{1-q^{\gamma_i-\gamma_j} t^{v_\gamma(i)-v_\gamma(j)-1}}{1-q^{\gamma_i-\gamma_j} t^{v_\gamma(i)-v_\gamma(j)}}\right)\right)$$

Classical world		qt World
x^μ		$E_\mu(q,t)$
$a_{\lambda+\rho}$	$\leftrightsquigarrow$	$A_{\lambda+\rho}(q,t)$
m_λ		$P_\lambda(q,t)$

Schur function $\quad s_\lambda = \dfrac{a_{\lambda+\rho}}{a_\rho} \qquad\qquad P_\lambda(q,qt) = \dfrac{A_{\lambda+\rho}(q,t)}{A_\rho(q,t)}$