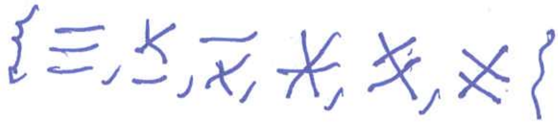
16.04.2026 (1)

<u>Combinatorial Representation Theory</u> Rep. Theory Seminar

A. Ram

The symmetric group S_3 is

$S_3 =$ 

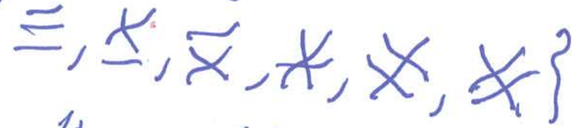
with product given by concatenation of diagrams

Let $s_1 =$  and $s_2 =$ . Then

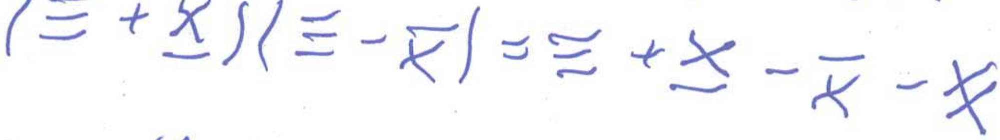
$s_1^2 = \equiv$ and $s_2^2 = \equiv$ and $s_1 s_2 s_1 = s_2 s_1 s_2$.

Representation theory is the art of representing algebraic structures by matrices.

Matrices means linear algebra

$\mathbb{C}S_3 = \mathrm{span}$ 

is a vector space with multiplication (an algebra)



<u>Theorem</u> (Artin-Wedderburn)

Every decent algebra is isomorphic to block diagonal matrices.

<u>Example</u> $\mathbb{C}S_3 \xrightarrow{\sim} \left\{ \begin{pmatrix} a &&& \\ & b & c & \\ & d & e & \\ &&& f \end{pmatrix} \;\middle|\; a,b,c,d,e,f \in \mathbb{C} \right\}$

$\Xi \mapsto \begin{pmatrix} 1 &&& \\ & 1 & 0 & \\ & 0 & 1 & \\ &&& 1 \end{pmatrix}$ $\quad \underline{X} \mapsto \begin{pmatrix} 1 &&& \\ & 1 & 0 & \\ & 0 & -1 & \\ &&& -1 \end{pmatrix}$ $\quad \overline{X} \mapsto \begin{pmatrix} 1 &&& \\ & \frac{1}{2} & \frac{3}{2} & \\ & \frac{1}{2} & -\frac{1}{2} & \\ &&& -1 \end{pmatrix}$

$P = \frac{1}{6}(\Xi + \underline{X} + \overline{X} + \ast + \cancel{X} + \bcancel{X}) \mapsto \begin{pmatrix} 1 &&& \\ & 0 & 0 & \\ & 0 & 0 & \\ &&& 0 \end{pmatrix}$

$N = \frac{1}{6}(\Xi - \underline{X} - \overline{X} - \ast + \cancel{X} + \bcancel{X}) \mapsto \begin{pmatrix} 0 &&& \\ & 0 & 0 & \\ & 0 & 0 & \\ &&& 1 \end{pmatrix}$

$E_{11} = \frac{1}{2}(\Xi + \underline{X}) - P \mapsto \begin{pmatrix} 0 &&& \\ & 1 & 0 & \\ & 0 & 0 & \\ &&& 0 \end{pmatrix}$

$E_{22} = \frac{1}{2}(\Xi - \underline{X}) - N \mapsto \begin{pmatrix} 0 &&& \\ & 0 & 0 & \\ & 0 & 1 & \\ &&& 0 \end{pmatrix}$

$\frac{2}{3} E_{11} \overline{X} E_{22} \mapsto \begin{pmatrix} 0 &&& \\ & 0 & 1 & \\ & 0 & 0 & \\ &&& 0 \end{pmatrix}$ $\quad 2 E_{22} \overline{X} E_{11} \mapsto \begin{pmatrix} 0 &&& \\ & 0 & 0 & \\ & 1 & 0 & \\ &&& 0 \end{pmatrix}$

<u>Definition</u> A decent algebra is an algebra isomorphic to block diagonal matrices.

<u>Theorem</u> If G is a finite group then $\mathbb{C}G$ is a decent algebra.

$$\mathbb{C}S_4 \xrightarrow{\sim} \begin{pmatrix} * &&&&&&&&& \\ & * & * & * &&&&&& \\ & * & * & * &&&&&& \\ & * & * & * &&&&&& \\ &&&& * & * &&&& \\ &&&& * & * &&&& \\ &&&&&& * & * & * & \\ &&&&&& * & * & * & \\ &&&&&& * & * & * & \\ &&&&&&&&& * \end{pmatrix}$$

Algebras of functions

The ~~set~~ algebra of functions $f: \{1,2\} \to \{1,2\}$ is

$$\mathbb{C}F_2 = \mathrm{span}\{=, \times, \underset{\cdot}{<}, \overset{\cdot}{\angle}\} \quad \text{and} \quad \mathbb{C}F_2^{\le} = \mathrm{span}\{=, \underset{\cdot}{<}, \overset{\cdot}{\angle}\}$$

is the ~~set~~ algebra of functions $f: \{1,2\} \to \{1,2\}$ such that if $i \le j$ then $f(i) \le f(j)$.

$\mathbb{C}F_2^{\le}$ and $\mathbb{C}F_2$ are <u>not</u> commutative

$$\underset{\cdot}{<}\,\overset{\cdot}{\angle} = \underset{\cdot}{<} \quad \text{and} \quad \overset{\cdot}{\angle}\,\underset{\cdot}{<} = \overset{\cdot}{\angle}.$$

So
$$\mathbb{C}F_2^{\le} \not\cong \left\{ \begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix} \,\middle|\, a,b,c \in \mathbb{C} \right\}.$$

So, by dimension counts, $\mathbb{C}F_2^{\le}$ is <u>not</u> a decent algebra

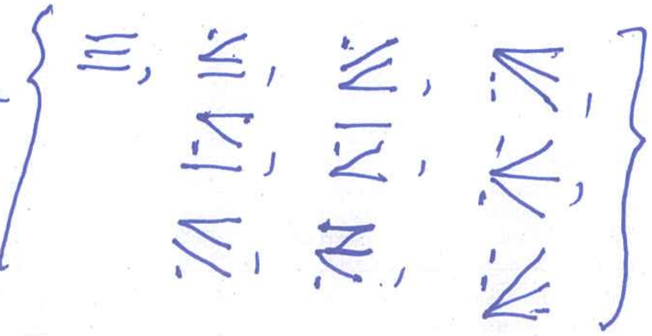
<u>Theorem</u> $\mathbb{C}F_2^{\le} \xrightarrow{\sim} \left\{ \begin{pmatrix} a & 0 \\ c & b \end{pmatrix} \,\middle|\, a,b,c \in \mathbb{C} \right\}$

$$\underset{\cdot}{<} \mapsto \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \quad \overset{\cdot}{\angle} \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \quad = \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\tfrac{1}{2}(\overset{\cdot}{\angle} + \underset{\cdot}{<}) \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \tfrac{1}{2}(\overset{\cdot}{\angle} - \underset{\cdot}{<}) \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$= -\tfrac{1}{2}(\underset{\cdot}{<} + \overset{\cdot}{\angle}) \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

<u>Remark</u> Let $r = \overset{\cdot}{\angle} - \underset{\cdot}{<}$ and $R = \mathrm{span}\{r\}$.
Then R is an ideal in $\mathbb{C}F_2^{\le}$ (and $\mathbb{C}F_2$) and all elements of R are nilpotent.
(i.e. some power of the element is 0).

$\mathbb{C}F_3^{\leq} = \mathrm{span}$ 

$= \mathrm{span}\left\{ \begin{matrix} 123, 223, & 233, & 111 \\ 113, & 133, & 222 \\ 112, & 122, & 333 \end{matrix} \right\}$

Nora's amazing observation Let

$n_1 =$  $+$ $- \equiv$ and $n_2 =$  $+$  $- \equiv$.

Then

$n_1^2 = \equiv$ and $n_2^2 = \equiv$ and $n_1 n_2 n_1 = n_2 n_1 n_2$.

So there is an image of $\mathbb{C}S_3$ in $\mathbb{C}F_3^{\leq}$!!!!

Questions

(1) Is the map $S_n \to \mathbb{C}F_n^{\leq}$ injective? (faithful representation)

Nora says yes and I agree

Is the map $\mathbb{C}S_n \to \mathbb{C}F_n^{\leq}$ injective? No.

(2) $\mathbb{C}F_n^{\leq}$ and $\mathbb{C}F_n$ are $\mathbb{C}S_n$-modules (vector spaces with a $\mathbb{C}S_n$-action).
Find their decomposition into irreducible pieces.

(3) Describe the largest ideal R in $\mathbb{C}F_n^{\leq}$ (and/or $\mathbb{C}F_n$) such that all elements are nilpotent.

(4) R is a $\mathbb{C}S_n$-module.
What is it's decomposition into $\mathbb{C}S_n$-modules?

(5) $F_n^{\leq}$ and F_n are monoids.
Can these monoid be efficiently described with generators and relations?

(6) For $w \in S_n$ let n_w be the corresponding element in $\mathbb{C}F_n^{\leq}$,

$$n_w = \sum_{f \in F_n^{\leq}} c_{wf} f, \quad \text{with } c_{wf} \in \mathbb{Z}.$$

Describe c_{wf} in terms of w and f.

Nora thinks $c_{wf} \in \{0, 1, -1\}$. I think not.
Try $w = 9467182350$ in S_{10} and/or
$w = c810d942fa53b6e7$

(7) What are the irreducible representations of $\mathbb{C}F_n$ and $\mathbb{C}F_n^{\leq}$?

See Hewitt and Zuckerman, Illinois J Math $\underline{1}$, (1957), 188-213.

and

Ganyushkin and Mazorchuk, Classical Finite Transformation Semigroups, Springer 2009