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Row reduction and Inverses

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Example Find the inverse of $A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & -1 & 1 \\ 0 & 1 & 3 \end{pmatrix}$.

Solution: Form the augmented matrix $[A | I_3]$ and perform row operations so that A is reduced to row echelon form.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right] \quad R_2 \to R_2 + R_1$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right] \quad R_3 \to R_3 - R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 - R_3 \\ R_2 \to R_2 - 2R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 2 & 1 & -1 \\ 0 & 1 & 0 & 3 & 3 & -2 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right] \quad R_1 \to R_1 - 2R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & -5 & 3 \\ 0 & 1 & 0 & 3 & 3 & -2 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$$

$$A^{-1} = \begin{pmatrix} -4 & -5 & 3 \\ 3 & 3 & -2 \\ -1 & -1 & 1 \end{pmatrix}$$

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<u>Rewrite</u>: Find the inverse of $A=\begin{pmatrix}1&2&1\\-1&-1&1\\0&1&3\end{pmatrix}$

<u>Solution</u>: Start with $AA^{-1}=1$ which is

$$\begin{pmatrix}1&2&1\\-1&-1&1\\0&1&3\end{pmatrix}A^{-1}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

Left multiply both sides by

$$\begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix} \text{ to get } \begin{pmatrix}1&2&1\\0&1&2\\0&1&3\end{pmatrix}A^{-1}=\begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix}$$

Left multiply both sides by

$$\begin{pmatrix}1&0&0\\0&1&0\\0&-1&1\end{pmatrix} \text{ to get } \begin{pmatrix}1&2&1\\0&1&2\\0&0&1\end{pmatrix}A^{-1}=\begin{pmatrix}1&0&0\\1&1&0\\-1&-1&1\end{pmatrix}$$

Left multiply both sides by

$$\begin{pmatrix}1&0&-1\\0&1&-2\\0&0&1\end{pmatrix} \text{ to get } \begin{pmatrix}1&2&0\\0&1&0\\0&0&1\end{pmatrix}A^{-1}=\begin{pmatrix}2&1&-1\\3&3&-2\\1&-1&1\end{pmatrix}$$

Left multiply both sides by

$$\begin{pmatrix}1&-2&0\\0&1&0\\0&0&1\end{pmatrix} \text{ to get } \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}A^{-1}=\begin{pmatrix}-4&-5&3\\3&3&-2\\-1&-1&1\end{pmatrix}$$

So

$$A^{-1}=\begin{pmatrix}-4&-5&3\\3&3&-2\\-1&-1&1\end{pmatrix}$$

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Solutions of linear systems

$$M_{t\times s}(\mathbb{Q}) = \{ t\times s \text{ matrices with entries in } \mathbb{Q}\}$$

$$GL_n(\mathbb{Q}) = \left\{ P\in M_{n\times n}(\mathbb{Q}) \;\middle|\; \begin{array}{l}\text{there exists } P^{-1}\in M_{n\times n}(\mathbb{Q}) \\ \text{with } PP^{-1}=1 \text{ and } P^{-1}P=1\end{array} \right\}$$

$$\mathbb{Q}^s = M_{s\times 1}(\mathbb{Q}).$$

For $A\in M_{t\times s}(\mathbb{Q})$ and $b\in \mathbb{Q}^t$ define

$$\mathrm{Sol}(Ax=b) = \{ x\in\mathbb{Q}^s \mid Ax=b\}$$

<u>Theorem</u> Let $P\in GL_t(\mathbb{Q})$, $Q\in GL_s(\mathbb{Q})$, $b\in\mathbb{Q}^t$, $r\in\{1,\ldots,\min(s,t)\}$. Let

$$1_r = E_{11}+\cdots+E_{rr} \text{ in } M_{t\times s}(\mathbb{Q}), \quad \text{where}$$

E_{ij} has 1 in the (i,j) entry and 0 elsewhere.
Let

$$A = P 1_r Q.$$

Then

$$\mathrm{Sol}(Ax=b) = \begin{cases} Q^{-1}\begin{pmatrix} (P^{-1}b)_1 \\ \vdots \\ (P^{-1}b)_r \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \mathrm{span}\left\{ \begin{array}{c}\text{last } s-r \\ \text{columns of } Q^{-1}\end{array}\right\}, & \begin{array}{l}\text{if the last } t-r \text{ entries} \\ \text{of } P^{-1}b \text{ are } 0,\end{array} \\ \emptyset, & \text{otherwise.} \end{cases}$$

The row reduction theorem

Theorem Let $A \in M_{t\times s}(\mathbb{Q})$.

Then there exist $P \in GL_t(\mathbb{Q})$, $Q \in GL_s(\mathbb{Q})$ and $r \in \{1, \dots, \min(s,t)\}$ such that

$$A = P 1_r Q.$$

Alternatively,

$$M_{t\times s}(\mathbb{Q}) = \bigsqcup_{r=0}^{\min(s,t)} GL_t(\mathbb{Q})\, 1_r\, GL_s(\mathbb{Q}).$$

Proof For $i \in \{1, \dots, t-1\}$ and $c \in \mathbb{Q}$ let

$$s_i(c) = 1 - E_{ii} - E_{i+1,i+1} + E_{i,i+1} + E_{i+1,i} + cE_{ii}.$$

Then

$$s_i(c)^{-1} = 1 - E_{ii} - E_{i+1,i+1} + E_{i,i+1} + E_{i+1,i} - cE_{i+1,i+1}.$$

Let j_1 be maximal such that column j_1 of A has a nonzero entry.
Let i_1 be maximal such that $A(i_1, j_1) \neq 0$.
Let

$$A^{(1)} = s_1\left(\frac{A(1,j_1)}{A(i_1,j_1)}\right)^{-1} s_2\left(\frac{A(2,j_1)}{A(i_1,j_1)}\right)^{-1} \cdots s_{i_1-1}\left(\frac{A(i_1-1,j_1)}{A(i_1,j_1)}\right)^{-1} A$$

Let j_2 be minimal such that

column j_2 of $A^{(1)}$ has a nonzero entry below row 1

Let $i_2 > 1$ be maximal such that $A^{(1)}(i_2, j_2) \neq 0$.

Let

$$A^{(2)} = s_2\left(\frac{A^{(1)}(2,j_2)}{A^{(1)}(i_2,j_2)}\right)^{-1} s_3\left(\frac{A^{(1)}(3,j_2)}{A^{(1)}(i_2,j_2)}\right)^{-1} \cdots s_{i_2-1}\left(\frac{A^{(1)}(i_2-1,j_2)}{A^{(1)}(i_2,j_2)}\right)^{-1} A^{(1)}$$

Let j_3 be minimal such that

column j_3 of $A^{(2)}$ has a nonzero entry below row 2.

Let $i_3 > 2$ be maximal such that $A^{(2)}(i_3, j_3) \neq 0$.

Let

$$A^{(3)} = s_3\left(\frac{A^{(2)}(3,j_3)}{A^{(2)}(i_3,j_3)}\right)^{-1} s_4\left(\frac{A^{(2)}(4,j_3)}{A^{(2)}(i_3,j_3)}\right)^{-1} \cdots s_{i_3-1}\left(\frac{A^{(2)}(i_3-1,j_3)}{A^{(2)}(i_3,j_3)}\right)^{-1} A^{(2)}$$

Continue this process until it happens that there does not exist j_{r+1} such that

column j_{r+1} of $A^{(r)}$ has a nonzero entry below row r.

Then

$$A = \left(s_{i_1-1}\left(\frac{A(i_1-1,j_1)}{A(i_1,j_1)}\right) \cdots s_2\left(\frac{A(2,j_1)}{A(i_1,j_1)}\right) s_1\left(\frac{A(1,j_1)}{A(i_1,j_1)}\right)\right)$$

$$\cdots \left(s_{i_r-1}\left(\frac{A^{(r-1)}(i_r-1,j_r)}{A^{(r-1)}(i_r,j_r)}\right) \cdots s_r\left(\frac{A^{(r-1)}(r,j_r)}{A^{(r-1)}(i_r,j_r)}\right)\right) A^{(r)}$$

where $A^{(r)}$ has the property that

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the first nonzero entry in row $k+1$ is to the right of the first nonzero entry in row k.

Let $h_i(d) = 1 + (d-1)E_{ii}$ and

$$x_{ij}(c) = 1 + cE_{ij}$$

Then

$$A^{(r)} = h_1(A^{(r)}(1,j_1)) \cdots h_r(A^{(r)}(r,j_r))$$
$$\cdot \left(x_{r-1,j_r}\left(\frac{A^{(r)}(r-1,j_r)}{A^{(r)}(r-1,j_{r-1})}\right) \cdots x_{1,j_r}\left(\frac{A^{(r)}(1,j_r)}{A^{(r)}(1,j_1)}\right)\right)$$
$$\cdots$$
$$\cdot \left(x_{2,j_3}\left(\frac{A^{(r)}(2,j_3)}{A^{(r)}(2,j_2)}\right) x_{1,j_3}\left(\frac{A^{(r)}(1,j_3)}{A^{(r)}(1,j_1)}\right)\right)$$
$$\cdot x_{1,j_2}\left(\frac{A^{(r)}(1,j_2)}{A^{(r)}(1,j_1)}\right) \cdot 1_r \cdot Q$$

where $Q \in GL_s(\mathbb{Q})$.