<u>Theorem</u> $\mathbb{C}$ is algebraically closed.

<u>Theorem</u> If $n\in\mathbb{Z}_{>0}$ and $a_0, a_1, \ldots, a_{n-1}\in\mathbb{C}$ and $a_0\neq 0$ and

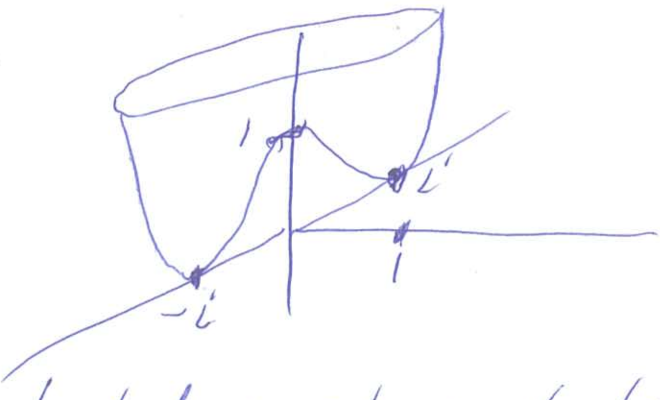
$$p(z)=z^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0$$

then there exists $\alpha\in\mathbb{C}$ such that $p(\alpha)=0$.

Define $d:\mathbb{C}\to\mathbb{R}_{\geq 0}$ by

$$d(z)=|p(z)|.$$

<u>Example</u> If $p(z)=z^2+1$ then the graph of d is



<u>Proposition 1</u> Let $a_n=1$ and let

$$S=|a_0|+|a_1|+\cdots+|a_n|$$

If $z\in\mathbb{C}$ and $|z|>|a_0|+S$ then $d(z)>|a_0|$

<u>Proposition 2</u> There exists $y\in\mathbb{C}$ with $|y|<1$, such that

$$d(y)<|a_0|.$$

Proof of the theorem

Let $N = S+7$ and

$$\overline{B_N(0)} = \{ z \in \mathbb{C} \mid |z| \le N \}$$

By the intermediate value theorem, there exist $l, m \in \mathbb{R}_{\ge 0}$ such that

$$d(\overline{B_N(0)}) = \mathbb{R}_{[l,m]}.$$

By Proposition 1,

$$d(\overline{B_N(0)} - \overline{B_S(0)}) \subseteq \mathbb{R}_{(|a_0|, m]}$$

So there exists $\alpha \in B_S(0)$ such that

$$d(\alpha) = l \quad \text{where} \quad l = \min\{ d(z) \mid z \in B_S(0) \}$$

Let

$$q(z) = p(z+\alpha).$$

Then $|q(0)| = |p(\alpha)| = l$ and

$$q(z) = z^n + b_{n-1} z^{n-1} + \cdots + b_1 z + b_0 \quad \text{with } |b_0| = l.$$

Case $l = 0$: Then $q(z) = z^n + b_{n-1} z^{n-1} + \cdots + b_1 z$ and $p(\alpha) = q(0) = 0$.

Case $l \ne 0$: By Proposition 2, this case cannot occur since if $l \ne 0$ then

$$\min\{ |q(y)| \mid |y| < 1 \} < l.$$

Proof of Proposition 2

Let $k\in\{1,\dots,n\}$ be minimal such that $a_k\neq 0$.

So
$$p(z)=z^n+a_{n-1}z^{n-1}+\cdots+a_kz^k+a_0.$$

Write
$$a_k=re^{i\beta}\quad\text{and}\quad a_0=\ell e^{i\theta}\quad(\text{so } r,\ell\in\mathbb{R}_{>0}).$$

Let
$$B=|a_n|+\cdots+|a_{k+1}|=5-\ell-r,$$
$$G=1+\left(\frac{\ell}{r}\right)^{1/k}+\left(\frac{\ell}{r}\right)^{1/k}\frac{B}{r},$$
$$y=\frac{1}{G}\left(\frac{\ell}{r}e^{i(\pi-\beta+\theta)}\right)^{1/k}=\frac{1}{G}\left(-a_0a_k^{-1}\right)^{1/k}$$

so that
$$y^k=\frac{-1}{G^k}a_0a_k^{-1}\quad\text{and}\quad |y|=\frac{1}{G}\left(\frac{\ell}{r}\right)^{1/k}<1.$$

Then
$$|p(y)|=|a_ny^n+\cdots+a_{k+1}y^{k+1}+a_ky^k+a_0|$$
$$\le |a_ny^n|+\cdots+|a_{k+1}y^{k+1}|+|a_ky^k+a_0|$$
$$=|y|^{k+1}\left(|a_n||y|^{n-(k+1)}+\cdots+|a_{k+1}|\right)+|a_0|\cdot|a_ka_0^{-1}y^k+1|$$
$$=\frac{1}{G^{k+1}}\frac{\ell}{r}\left(\frac{\ell}{r}\right)^{1/k}B+\ell\cdot\left(\frac{-1}{G^k}+1\right)$$
$$=\frac{\ell}{G^k}\,\frac{\left(\frac{\ell}{r}\right)^{1/k}\frac{B}{r}}{G}+\ell-\frac{\ell}{G^k}<\frac{\ell}{G^k}+\ell-\frac{\ell}{G^k}=\ell.$$

Proof of Proposition 1

$$S = |a_0| + |a_1| + \cdots + |a_n|$$

Assume $z \in \mathbb{C}$ and $|z| > |a_0| + S$.

Since
$$|z^n| = |p(z) - (a_{n-1} z^{n-1} + \cdots + a_0)|$$
$$\leq |p(z)| + |a_{n-1} z^{n-1} + \cdots + a_0|$$
$$\leq |p(z)| + |a_{n-1}| |z|^{n-1} + \cdots + |a_0| \cdot |z|^{n-1}$$
$$= |p(z)| + |z|^{n-1} (S-1)$$

then
$$|p(z)| \geq |z|^n - |z|^{n-1} (S-1)$$
$$= |z|^{n-1} (|z| - (S-1))$$
$$> (|a_0| + S)^{n-1} \cdot (|a_0| + S - S + 1)$$
$$> 1^{n-1} \cdot |a_0| = |a_0|.$$