On a conjecture of H.O. Foulkes

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Abstract

Alain Lascoux and Marcel-Paul Schützenberger, *Sur une conjecture de H.O. Foulkes*, C.R. Acad.Sc.Paris, Série A - 323, **286** (20 février 1978). Translated from the French by Arun Ram.

We announce a proof of a conjecture of H.O. Foulkes on certain polynomials arising in the symmetric functions associated to projective representations of the symmetric group and the linear groups over finite fields.

One sketches a proof of Foulkes' conjecture on the polynomials defining Littlewood Q-functions in terms of Schur functions.

Key words— Tableaux, charge, Kostka-Foulkes polynomials

Let $\mathbb{Z}^N$ denote the set of maps I from $\mathbb{N}$ to $\mathbb{Z}$ such that $nI = 0$ for all n large enough to define $I^\Sigma \in \mathbb{Z}^N$ for $nI^\Sigma = \sum_{m \geq n} mI$; the *weights* of I are thus $0I^\Sigma$. The partitions of an integer n are the $I \in \mathbb{Z}^N$ of weight n such that $0I \geq 1I \geq 2I \geq \cdots$.

Littlewood [Li61] has defined a basic family $\{\mathbb{Q}(I)\}$ of symmetric functions (in the variables of an arbitrary set which is unnecessary to make explicit) indexed by the partitions, via the identity

$$\mathbb{Q}(I) = \sum s'(J)F(I;J), \tag{1}$$

in which the $s'(J)$ are the Schur functions (modified), the sum is understood to be over all partition J of the same weight as I and the $F(I;J)$ are the polynomies with integer coefficients and a new variable q. We propose to call these last polynomials *Foulkes polynomials* in memory of H.O. Foulkes to whom are many beautiful results on symmetric functions which has produced [Fo74] the conjecture that all their coefficients are in $\mathbb{N}$. We remark that these polynomials are the Euler-Poincaré polynomials of line bundles on flag varieties.

We announce the:

Theorem 0.1. *$F(I;J)$ is a monic polynomial with nonnegative coefficients which is 0 if one of the differences $I^\Sigma - nJ^\Sigma$ ($n \in \mathbb{N}$) is negative and otherwise with degree equal to their sum.*

The proof uses Young tableaux. Let A^* the free onoide generated by a totally ordered alphabet A and $\equiv$ the smallest congruence satisfies $xzy \equiv zxy$ and $zty \equiv tzy$ for all letters x, y, z, t of A such that $x \leq y < z \leq t$.

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There exists a section $A^*/\equiv\to A^*$; we say that its image is the set of Young tableaux. The composition $A^*\to A^*/\equiv\to T$ is the *recovery*. T contains the subset L of products $u_k u_{k-1}\dots u_0$ of rows such that the degrees $|u_0|\ge|u_1|\ge\cdots\ge|u_k|$ form a partition $\|t\|$ (called the *shape* of t).

Let $A^*\uparrow=\bigcup A^{*I}$ the submodule of words w of A such that the multidegree is a partition I (of their degree $|w|$). We introduce a map $\nu\colon A^*\uparrow\to\mathbb{N}$ (the *charge*) satisfies the two conditions:

1. $f,g\in A^*\uparrow$, $f\equiv g\Rightarrow f\nu=g\nu$;

2. $f,g\in A^*$, $fg\in A^*\uparrow$, the degree $|g|_a$ of g in a (the first letter of A) is $0\Rightarrow(fg)\nu=(gf)\nu+|g|$.

The full definition requires another extremality condition too heavy to give here. If I,J are two partitions of the same weight, one knows that the set $T^I\cap A^{*I}$ of tableaux of shape J and of multidergee I is empty unless the differences $nI^\Sigma-nJ^\Sigma$ are all nonnegative, and we show that the charge $t\nu$ is at most equal to $O(I^\Sigma-J^\Sigma)^\Sigma$, the equality is attained for a unique tableau.

In consequence, Theorem 0.1 gives:

Theorem 0.2. *Let I and J be two partitions of the same weight. The Foulkes polynomial $F(I,J)$ is equal to $\sum q^{tV}$, sum over all the tableaux $t\in T^J\cap A^{*I}$.*

This result contains earlier results of Macdonald [*cf.* [Mor77]], and a student of Foulkes, Thomas [Th77].

Thanks to an induction lemma due to Morris [Mor63], and to the Pieri formula for multiplication of a tableau by a row, we reduce the expression (1) of $Q(I)$:

$$Q(I)=\sum\{s'(\|tu\|)q^{|u|-0I}q^{(ut)\nu}\mid u\in L,t\in T,|t|_a=0,ut\in A^{*I}\}.\tag{2}$$

In this equation $\|tu\|$ denotes the element J of $\mathbb{Z}^{\mathbb{N}}$ such that

$$0J=|u|;\quad 1J=0\|t\|,\quad \dots,\quad nJ=(n-1)\|t\|,\quad \dots$$

Following the classical conventions, $s'(J)=\mathrm{sgn}(J).s'(J^\Lambda)$ where $\mathrm{sgn}J\in\{-1,0,1\}$ and where, when $\mathrm{sgn}J\neq0$, J^Λ is a certain partition of the same weight as J.

The charge has the property that $(|u|-0I)+(ut)v=(tu)v$, to establish Theorem 0.2 it suffices that the restriction of the sum (2) to the set X of pairs (t,u) such that $tu\notin T$ and $\mathrm{sgn}(\|tu\|)\neq0$. This is the core of the proof, and relies on the exact sequence introduced in [Las77] (p. O.64). We show that there exists a subset X_- of X and a bijection ζ from X_- to $X\backslash X_-$, such that if $(t,u)\zeta=(t',u')$ then $tu\equiv t'u'$, whence $(tu)v=(t'u')v$ and $s'(\|tu\|)+s'(\|t'u'\|)=0$. The details will be published soon.

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