Remark 1.3 *(The source of the statistics $v_\mu(r)$ and $u_\mu(r,c)$).* The minimal length permutation which rearranges μ into weakly increasing order is $v_\mu = (v_\mu(1), \ldots, v_\mu(n))$. The affine Weyl group for type GL_n is the group of n-periodic permutations. If t_μ denotes the n-periodic permutation which is the translation in μ then $t_\mu = u_\mu v_\mu$ with $\ell(t_\mu) = \ell(u_\mu) + \ell(v_\mu)$ and u_μ has a reduced word

$$u_\mu = \prod_{(r,c)\in\mu} (s_{u_\mu(r,c)} \cdots s_2 s_1 \pi),$$

where $s_i \in S_n$ is the transposition which switches i and $i+1$ and π is the n-periodic permutation given by $\pi(i) = i+1$. See [9, §2 and Prop. 2.2(a)]. $\square$

2. Operator expansions

Let $j \in \{1, \ldots, n\}$ and let $C \subseteq \{1, \ldots, n\}$. Writing $C = \{a_1, \ldots, \underset{m}{\cancel{a_n}}\}$ with $a_1 < \cdots < a_m$ define

$$f_C(Y) = \frac{t^{-(m-1)/2}}{1 - qY_{a_1}Y_{a_m}^{-1}} \Big(\prod_{i=1}^{m-1} \frac{1-t}{1 - Y_{a_i}Y_{a_{i+1}}^{-1}} \Big). \tag{2.1}$$

Then define

$$F_{C,j}(Y) = \begin{cases} 0, & \text{if } j \notin C, \\ 1 - qY_{a_1}Y_{a_m}^{-1}, & \text{if } j = a_p \text{ and } p = 1, \\ Y_{a_1}Y_{a_p}^{-1} - Y_{a_1}Y_{a_{p-1}}^{-1}, & \text{if } j = a_p \text{ and } p \neq 1, \end{cases} \tag{2.2}$$

$$A_{C,j}(Y) = \begin{cases} 0, & \text{if } j < a_1, \\ Y_{a_1}Y_{a_p}^{-1} - qY_{a_1}Y_{a_m}^{-1}, & \text{if } a_p \leq j < a_{p+1}, \\ (1-q)Y_{a_1}Y_{a_m}^{-1}, & \text{if } j > a_m, \end{cases} \tag{2.3}$$

$$\Phi_{C,j}(Y) = \begin{cases} 0, & \text{if } j \notin C, \\ 1 - qY_{a_1}Y_{a_m}^{-1}, & \text{if } j = a_p \text{ and } p = m, \\ Y_{a_1}Y_{\underset{a_{p+1}}{\cancel{a_p}}}^{-1} - Y_{a_1}Y_{\underset{a_p}{\cancel{a_{p-1}}}}^{-1}, & \text{if } j = a_p \text{ and } p \neq m, \end{cases} \tag{2.4}$$

$$\Psi_{C,j}(Y) = \begin{cases} 0, & \text{if } j \geq a_m, \\ Y_{a_p}Y_{a_m}^{-1} - qY_{a_1}Y_{a_m}^{-1}, & \text{if } a_{p-1} < j \leq a_p, \\ (1-q)Y_{a_1}Y_{a_m}^{-1}, & \text{if } j \leq a_1, \end{cases} \tag{2.5}$$

and

$$\begin{aligned} B_{C,j}(Y) &= F_{C,j}(Y) + \frac{1-t}{1-qt^{n-j+1}} A_{C,j}(Y) \quad \text{and} \\ \Omega_{C,j}(Y) &= \Phi_{C,j}(Y) + \Big(\frac{1-t}{1-qt^j}\Big)\Psi_{C,j+1}(Y). \end{aligned} \tag{2.6}$$

For $k \in \{1, \dots, n\}$ such that $k \notin C$ define

$$b(k) = \begin{cases} \mu_{a_m} + 1, & \text{if } 1 \le k < a_1, \\ \mu_{a_i}, & \text{if } a_i < k < a_{i+1}, \\ \mu_{a_m}, & \text{if } a_m < k \le n, \end{cases} \quad \text{and} \quad c(k) = \begin{cases} v_{a_m^{\uparrow}\mu}(a_m), & \text{if } 1 \le k < a_1, \\ v_\mu(a_i), & \text{if } a_i < k < a_{i+1}, \\ v_\mu(a_m), & \text{if } a_m < k \le n, \end{cases} \tag{3.3}$$

and

$$d(k) = \begin{cases} \mu_{a_1} - 1, & \text{if } a_m < k \le n, \\ \mu_{a_i}, & \text{if } a_{i-1} < k < a_i, \\ \mu_{a_1}, & \text{if } 1 \le k < a_1, \end{cases} \quad \text{and} \quad e(k) = \begin{cases} v_{a_1^{\downarrow}\mu}(a_1), & \text{if } a_m < k \le n, \\ v_\mu(a_i), & \text{if } a_{i-1} < k < a_i, \\ v_\mu(a_1), & \text{if } 1 \le k < a_1. \end{cases}$$

Keeping $k \in \{1, \dots, n\}$ such that $k \notin C$ define

$$\mathrm{wt}_\mu(C, k) = \begin{cases} 0, & \text{if } b(k) = \mu_k, \\ 1, & \text{if } b(k) > \mu_k, \\ t\dfrac{(1 - q^{\mu_k - b(k)} t^{v_\mu(k) - c(k) + 1})(1 - q^{\mu_k - b(k)} t^{v_\mu(k) - c(k) - 1})}{(1 - q^{\mu_k - b(k)} t^{v_\mu(k) - c(k)})^2}, & \text{if } b(k) < \mu_k, \end{cases}$$

and

$$\mathrm{rwt}_\mu(C, k) = \begin{cases} 0, & \text{if } d(k) = \mu_k, \\ \cancel{t^{-1}},\ 1 & \text{if } \cancel{d(k) > \mu_k},\ d(k) < \mu_k \\ t\,\dfrac{(1 - q^{\mu_k - d(k)} t^{v_\mu(k) - e(k) + 1})(1 - q^{\mu_k - d(k)} t^{v_\mu(k) - e(k) - 1})}{(1 - q^{\mu_k - d(k)} t^{v_\mu(k) - e(k)})^2}, & \text{if } \cancel{d(k) < \mu_k}.\ d(k) > \mu_k \end{cases}$$

For $k \in \{1, \dots, n\}$ such that $k \in C$ define

$$\mathrm{wt}_\mu(C, k) = \mathrm{rwt}_\mu(C, k) = \begin{cases} \dfrac{1 - t}{1 - q^{\mu_{a_{i+1}} - \mu_{a_i}} t^{v_\mu(a_{i+1}) - v_\mu(a_i)}}, & \text{if } k = a_i \text{ and } i \ne m, \\ \dfrac{1}{1 - q^{\mu_{a_m} - \mu_{a_1} + 1} t^{v_\mu(a_m) - v_\mu(a_1)}}, & \text{if } k = a_m. \end{cases} \tag{3.4}$$

Then define

$$\mathrm{wt}_\mu(C) = t^{-\#\{i \mid \mu_i > \mu_{a_m}\}} \prod_{\cancel{i}\,k=1}^{n} \mathrm{wt}_\mu(C, k), \tag{3.5}$$

and

$$\mathrm{rwt}_\mu(C) = t^{\cancel{\#\{i \mid \mu_i > \mu_{a_1}\}}\ -\#\{i \mid \mu_i < \mu_{a_1}\}} \prod_{\cancel{i}\,k=1}^{n} \mathrm{rwt}_\mu(C, k), \tag{3.6}$$

Total correction factor for $\mathrm{rwt}_\mu(C)$: $t^{n-m-\#\{i \mid \mu_i > \mu_{a_1}\} - \#\{i \mid \mu_i < \mu_{a_1}\}} = t^{\#\{i \mid \mu_i = \mu_{a_1}\} - m}$

Theorem 3.1 *(Monk rules for Macdonald polynomials). Let $j \in \{1, \dots, n\}$ and $\mu \in \mathbb{Z}^n_{\geq 0}$. Let E_μ denote the electronic Macdonald polynomial indexed by μ in $\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Let*

$$F_\mu(C, j) = \begin{cases} 0, & \text{if } j \notin C. \\ 1 - q^{\mu_{a_m} - \mu_{a_1} + 1} t^{v_\mu(a_m) - v_\mu(a_1)}, & \text{if } j = a_p \text{ and } p = 1, \\ q^{\mu_{a_p} - \mu_{a_1}} t^{v_\mu(a_p) - v_\mu(a_1)} - q^{\mu_{a_{p-1}} - \mu_{a_1}} t^{v_\mu(a_{p-1}) - v_\mu(a_1)}, & \text{if } j = a_p \text{ and } p \neq 1, \end{cases}$$

$$A_\mu(C, j) = \begin{cases} 0, & \text{if } j < a_1, \\ q^{\mu_{a_p} - \mu_{a_1}} t^{v_\mu(a_p) - v_\mu(a_1)} - q^{\mu_{a_m} - \mu_{a_1} + 1} t^{v_\mu(a_m) - v_\mu(a_1)}, & \text{if } a_p \leq j < a_{p+1}, \\ (1 - q) q^{\mu_{a_m} - \mu_{a_1}} t^{v_\mu(a_m) - v_\mu(a_1)}, & \text{if } \cancel{j > a_m}, \quad j \geq a_m \end{cases}$$

and

$$B_\mu(C, j) = F_\mu(C, j) + \frac{1 - t}{1 - q t^{n-j+1}} A_\mu(C, j). \tag{3.7}$$

Let

$$\Phi_\mu(C, j) = \begin{cases} 0, & \text{if } j \notin C. \\ 1 - q^{\mu_{a_m} - \mu_{a_1} + 1} t^{v_\mu(a_m) - v_\mu(a_1)}, & \text{if } j = a_p \text{ and } p = m, \\ q^{\mu_{\underset{a_{p+1}}{\cancel{a_p}}} - \mu_{a_1}} t^{v_\mu(\underset{a_{p+1}}{\cancel{a_p}}) - v_\mu(a_1)} - q^{\mu_{\underset{a_p}{\cancel{a_{p-1}}}} - \mu_{a_1}} t^{v_\mu(\underset{a_p}{\cancel{a_{p-1}}}) - v_\mu(a_1)}, & \text{if } j = a_p \text{ and } p \neq m, \end{cases}$$

$$\Psi_\mu(C, j) = \begin{cases} 0, & \text{if } \cancel{j < a_1}, \quad j > a_m \\ q^{\mu_{a_m} - \mu_{a_p}} t^{v_\mu(a_m) - v_\mu(a_p)} - q^{\mu_{a_m} - \mu_{a_1} + 1} t^{v_\mu(a_m) - v_\mu(a_1)}, & \text{if } a_{p-1} < j \leq a_p, \\ (1 - q) q^{\mu_{a_m} - \mu_{a_1}} t^{v_\mu(a_m) - v_\mu(a_1)}, & \text{if } j \leq a_1, \end{cases}$$

and

$$\Omega_\mu(C, j) = \Phi_\mu(C, j) + \frac{1 - t}{1 - q t^j} \Psi_\mu(C, j). \tag{3.8}$$

Let $\mathrm{rot}_\mu(C)$ *and* $\mathrm{wt}_\mu(C)$ *as in* (3.1) *and* (3.5), *and let* $\mathrm{rrot}_\mu(C)$ *and* $\mathrm{rwt}_\mu(C)$ *be as defined in* (3.1) *and* (3.6). *Then*

(a)

$$x_j E_\mu = \sum_{\substack{C \subseteq \{1, \dots, n\} \\ C \cap \{j\} \neq \emptyset}} F_\mu(C, j) \mathrm{wt}_\mu(C) E_{\mathrm{rot}_C(\mu)},$$

(b)

$$(x_1 + \cdots + x_j) E_\mu = \sum_{\substack{C \subseteq \{1, \dots, n\} \\ C \cap \{1, \dots, j\} \neq \emptyset}} A_\mu(C, j) \mathrm{wt}_\mu(C) E_{\mathrm{rot}_C(\mu)},$$

(c)

$$E_{\varepsilon_j} E_\mu = \sum_{\substack{C \subseteq \{1, \dots, n\} \\ C \cap \{1, \dots, j\} \neq \emptyset}} B_\mu(C, j) \mathrm{wt}_\mu(C) E_{\mathrm{rot}_C(\mu)},$$

(d)

$$x_j^{-1}E_\mu = \sum_{\substack{C\subseteq\{1,\ldots,n\}\\ C\cap\{j\}\neq\emptyset}} \Phi_\mu(C,j)\mathrm{rwt}_\mu(C)E_{\mathrm{rrot}_C(\mu)},$$

(e)

$$(x_j^{-1}+\cdots+x_n^{-1})E_\mu = \sum_{\substack{C\subseteq\{1,\ldots,n\}\\ C\cap\{j,\ldots,n\}\neq\emptyset}} \Psi_\mu(C,j)\mathrm{rwt}_\mu(C)E_{\mathrm{rrot}_C(\mu)},$$

(f)

$$E_{-\varepsilon_j}E_\mu = \sum_{\substack{C\subseteq\{1,\ldots,n\}\\ C\cap\{j,\ldots,n\}\neq\emptyset}} \Omega_\mu(C,j)\mathrm{rwt}_\mu(C)E_{\mathrm{rrot}_C(\mu)}.$$

Proof. From [9, (4.1) and (4.2)], if $\mu_i > \mu_{i+1}$ then

$$\begin{aligned} &t^{\frac12}\tau_i^\vee E_\mu = E_{s_i\mu} \quad \text{and} \\ &t^{\frac12}\tau_i^\vee E_{s_i\mu} = t\frac{(1-q^{\mu_i-\mu_{i+1}}t^{v_\mu(i)-v_\mu(i+1)+1})(1-q^{\mu_i-\mu_{i+1}}t^{v_\mu(i)-v_\mu(i+1)-1})}{(1-q^{\mu_i-\mu_{i+1}}t^{v_\mu(i)-v_\mu(i+1)})^2}E_\mu. \end{aligned} \tag{3.9}$$

From [9, (3.5)],

$$\tau_\pi^\vee E_\mu = t^{\frac12(n-1)-\#\{i\in\{1,\ldots,n-1\}\ |\ \overset{\mu_i>\mu_n}{\cancel{\mu_i\leq\mu_n}}\}}E_{\pi\mu}. \tag{3.10}$$

If the complement of C in $\{1,\ldots n\}$ is

$$C^c = \{b_1,\ldots,b_{n-m}\} \quad \text{with} \quad b_1<\cdots<b_r<j<b_{r+1}<\cdots<b_{n-m}$$

then $\tau_{C,j}^\vee = \tau_{b_r}^\vee\tau_{b_{r-1}}^\vee\cdots\tau_{b_1}^\vee\tau_\pi^\vee\tau_{b_{n-1}-1}^\vee\cdots\tau_{b_{r+1}-1}^\vee$ and using (3.9) and (3.10) gives

$$\begin{aligned} t^{\frac12(n-m)}\tau_{C,j}^\vee E_\mu &= t^{\frac12}\tau_{b_r}^\vee t^{\frac12}\tau_{b_{r-1}}^\vee\cdots t^{\frac12}\tau_{b_1}^\vee\tau_\pi^\vee t^{\frac12}\tau_{b_{n-m}-1}^\vee\cdots t^{\frac12}\tau_{b_{r+1}-1}^\vee E_\mu \\ &= t^{\frac12(n-1)-\#\{\mu_i>\mu_{a_m}\}}\Big(\prod_{k\notin C}\mathrm{wt}_\mu(C,k)\Big)E_{\mathrm{rot}_C(\mu)}. \end{aligned} \tag{3.11}$$

An example of the step-by-step computation of $\tau_{C,j}^\vee E_\mu$ is given in Example 3.1.

Let $f_C(Y)$ and $F_{C,j}(Y)$ be as defined in (2.1) and (2.2), and let ev_μ^t be the evaluation map defined in (1.6). Since

$$\mathrm{ev}_\mu^t(Y_iY_j^{-1}) = q^{\mu_j-\mu_i}t^{v_\mu(j)-v_\mu(i)} \qquad \text{and} \qquad \mathrm{ev}_\mu^t\Big(\frac{1-t}{1-Y_iY_j^{-1}}\Big) = \frac{1-t}{1-q^{\mu_j-\mu_i}t^{v_\mu(j)-v_\mu(i)}}$$

then comparing (3.4) and (2.1) gives

$$\mathrm{ev}_\mu^t(f_C(Y)) = t^{-\frac12(m-1)}\prod_{k\in C}\mathrm{wt}_\mu(C,k) \qquad \text{and} \qquad \mathrm{ev}_\mu^t(F_{C,j}(Y)) = F_\mu(C,j). \tag{3.12}$$

Using (1.7) on the expression in Theorem 2.1(a) and inserting (3.12) and (3.11) gives

$$\begin{aligned}
x_jE_\mu = X_jE_\mu &= \sum_{\substack{C\subseteq\{1,\ldots,n\}\\ j\in C}} \tau^\vee_{C,j}F_{C.j}(Y)f_C(Y)E_\mu \\
&= \sum_{\substack{C\subseteq\{1,\ldots,n\}\\ j\in C}} \tau^\vee_{C,j}\mathrm{ev}^t_\mu(F_{C,j}(Y))\mathrm{ev}^t_\mu(f_C(Y))E_\mu \\
&= \sum_{\substack{C\subseteq\{1,\ldots,n\}\\ j\in C}} F_\mu(C,j)t^{-\frac12(m-1)}\Big(\prod_{k\in C}\mathrm{wt}_\mu(C,k)\Big)\tau^\vee_{C,j}E_\mu \\
&= \sum_{\substack{C\subseteq\{1,\ldots,n\}\\ j\in C}} F_\mu(C,j)t^{-\frac12(m-1)}\Big(\prod_{k\in C}\mathrm{wt}_\mu(C,k)\Big)t^{\frac12(n-1)-\#\{\overset{\mu_i>\mu_{a_m}}{\cancel{\mu_i<\mu_{a_m}}}\}} \\
&\quad\cdot\Big(\prod_{k\notin C}\mathrm{wt}_\mu(C,k)\Big)t^{-\frac12(n-m)}E_{\mathrm{rot}_C(\mu)} \\
&= \sum_{\substack{C\subseteq\{1,\ldots,n\}\\ j\in C}} F_\mu(C,j)\mathrm{wt}_\mu(C)E_{\mathrm{rot}_C(\mu)}.
\end{aligned}$$

This completes the proof of (a). The proof of the remaining parts is similar, using parts (b)–(f) of Theorem 2.1. □

Example 3.1 *(An example of the computation of $\tau^\vee_{C,j}E_\mu$).* Let $n = 11$ and $j = 7$ and

$$C = \{2,5,7,9,10\}, \qquad \text{so that} \quad \tau^\vee_{C,7} = \tau^\vee_6\tau^\vee_4\tau^\vee_3\tau^\vee_1\tau^\vee_\pi\tau^\vee_{10}\tau^\vee_7 = \tau^\vee_6\tau^\vee_4\tau^\vee_3\tau^\vee_1\tau^\vee_\pi\tau^\vee_{11-1}\tau^\vee_{8-1}$$

and $C^c = \{1,3,4,6,8,11\}$. Then, using (3.9) and (3.10),

$$\begin{aligned}
t^{\frac62}\tau^\vee_{C,7}E_\mu &= t^{\frac12}\tau^\vee_6t^{\frac12}\tau^\vee_4t^{\frac12}\tau^\vee_3t^{\frac12}\tau^\vee_1\tau^\vee_\pi t^{\frac12}\tau^\vee_{11-1}t^{\frac12}\tau^\vee_{8-1}E_{(\mu_1,\mu_2,\mu_3,\mu_4,\mu_5,\mu_6,\mu_7,\mu_8,\mu_9,\mu_{10},\mu_{11})} \\
&= \mathrm{wt}_\mu(C,8)t^{\frac12}\tau^\vee_6t^{\frac12}\tau^\vee_4t^{\frac12}\tau^\vee_3t^{\frac12}\tau^\vee_1\tau^\vee_\pi t^{\frac12}\tau^\vee_{11-1}E_{(\mu_1,\mu_2,\mu_3,\mu_4,\mu_5,\mu_6,\mu_8,\mu_7,\mu_9,\mu_{10},\mu_{11})} \\
&= \mathrm{wt}_\mu(C,8)\mathrm{wt}_\mu(C,11)t^{\frac12}\tau^\vee_6t^{\frac12}\tau^\vee_4t^{\frac12}\tau^\vee_3t^{\frac12}\tau^\vee_1\tau^\vee_\pi E_{(\mu_1,\mu_2,\mu_3,\mu_4,\mu_5,\mu_6,\mu_8,\mu_7,\mu_9,\mu_{11},\mu_{10})} \\
&= t^{\frac12(11-1)-\#\{\underset{\mu_i>\mu_{10}}{\cancel{\mu_i<\mu_{10}}}\}}\mathrm{wt}_\mu(C,8)\mathrm{wt}_\mu(C,11)t^{\frac12}\tau^\vee_6t^{\frac12}\tau^\vee_4t^{\frac12}\tau^\vee_3t^{\frac12}\tau^\vee_1 \\
&\quad\cdot E_{(\mu_{10}+1,\mu_1,\mu_2,\mu_3,\mu_4,\mu_5,\mu_6,\mu_8,\mu_7,\mu_9,\mu_{11})} \\
&= t^{5-\#\{\underset{\mu_i>\mu_{10}}{\cancel{\mu_i<\mu_{10}}}\}}\Big(\prod_{k\in\{1,8,11\}}\mathrm{wt}_\mu(C,k)\Big)t^{\frac12}\tau^\vee_6t^{\frac12}\tau^\vee_4t^{\frac12}\tau^\vee_3 \\
&\quad\cdot E_{(\mu_1,\mu_{10}+1,\mu_2,\mu_3,\mu_4,\mu_5,\mu_6,\mu_8,\mu_7,\mu_9,\mu_{11})} \\
&= t^{5-\#\{\underset{\mu_i>\mu_{10}}{\cancel{\mu_i<\mu_{10}}}\}}\Big(\prod_{k\in\{1,3,8,11\}}\mathrm{wt}_\mu(C,k)\Big)t^{\frac12}\tau^\vee_6t^{\frac12}\tau^\vee_4 \\
&\quad\cdot E_{(\mu_1,\mu_{10}+1,\mu_3,\mu_2,\mu_4,\mu_5,\mu_6,\mu_8,\mu_7,\mu_9,\mu_{11})}
\end{aligned}$$

$$
\begin{aligned}
&= t^{5-\#\{\underset{\mu_i>\mu_{10}}{\cancel{\mu_i<\mu_{10}}}\}}\Big(\prod_{k\in\{1,3,4,8,11\}} \mathrm{wt}_\mu(C,k)\Big)t^{\frac12}\tau_6^\vee \\
&\quad\cdot E_{(\mu_1,\mu_{10}+1,\mu_3,\mu_4,\mu_2,\mu_5,\mu_6,\mu_8,\mu_7,\mu_9,\mu_{11})} \\
&= t^{5-\#\{\underset{\mu_i>\mu_{10}}{\cancel{\mu_i<\mu_{10}}}\}}\Big(\prod_{k\in\{1,3,4,6,8,11\}} \mathrm{wt}_\mu(C,k)\Big)E_{(\mu_1,\mu_{10}+1,\mu_3,\mu_4,\mu_2,\mu_6,\mu_5,\mu_8,\mu_7,\mu_9,\mu_{11})} \\
&= t^{5-\#\{\underset{\mu_i>\mu_{10}}{\cancel{\mu_i<\mu_{10}}}\}}\Big(\prod_{k\in C^c} \mathrm{wt}_\mu(C,k)\Big)E_{\mathrm{rot}_C(\mu)}.
\end{aligned}
$$

The red entries correspond to the parts specified by C which are rotated to get $\mathrm{rot}_\mu(C)$ as in the picture in (3.2). □

Example 3.2 *(An example of the computation of $\rho_{D,j}E_\mu$).* Let $n = 11$ and $j = 7$ and

$$D = \{1,4,5,7,9\}, \qquad \text{so that} \quad \rho_{D,7} = \tau_7^\vee\tau_9^\vee\tau_{10}^\vee(\tau_\pi^\vee)^{-1}\tau_2^\vee\tau_3^\vee\tau_6^\vee$$

and $D^c = \{2,3,6,8,10,11\}$. Then, using (3.9) and (3.10),

$$
\begin{aligned}
t^{\frac62}\rho_{D,7}E_\mu &= t^{\frac12}\tau_7^\vee t^{\frac12}\tau_9^\vee t^{\frac12}\tau_{10}^\vee(\tau_\pi^\vee)^{-1}t^{\frac12}\tau_2^\vee t^{\frac12}\tau_3^\vee t^{\frac12}\tau_6^\vee E_{(\mu_1,\mu_2,\mu_3,\mu_4,\mu_5,\mu_6,\mu_7,\mu_8,\mu_9,\mu_{10},\mu_{11})} \\
&= \mathrm{rwt}_\mu(D,6)t^{\frac12}\tau_7^\vee t^{\frac12}\tau_9^\vee t^{\frac12}\tau_{10}^\vee(\tau_\pi^\vee)^{-1}t^{\frac12}\tau_2^\vee t^{\frac12}\tau_3^\vee E_{(\mu_1,\mu_2,\mu_3,\mu_4,\mu_5,\mu_7,\mu_6,\mu_8,\mu_9,\mu_{10},\mu_{11})} \\
&= \mathrm{rwt}_\mu(D,3)\mathrm{rwt}_\mu(D,6)t^{\frac12}\tau_7^\vee t^{\frac12}\tau_9^\vee t^{\frac12}\tau_{10}^\vee(\tau_\pi^\vee)^{-1}t^{\frac12}\tau_2^\vee \\
&\quad\cdot E_{(\mu_1,\mu_2,\mu_4,\mu_3,\mu_5,\mu_7,\mu_6,\mu_8,\mu_9,\mu_{10},\mu_{11})} \\
&= \Big(\prod_{k\in\{2,3,6\}} \mathrm{rwt}_\mu(D,k)\Big)t^{\frac12}\tau_7^\vee t^{\frac12}\tau_9^\vee t^{\frac12}\tau_{10}^\vee(\tau_\pi^\vee)^{-1} \\
&\quad\cdot E_{(\mu_1,\mu_4,\mu_2,\mu_3,\mu_5,\mu_7,\mu_6,\mu_8,\mu_9,\mu_{10},\mu_{11})} \\
&= t^{\overset{+}{-}\frac12(11-1)\overset{-}{+}\#\{\mu_i<\mu_1\}}\Big(\prod_{k\in\{2,3,6\}} \mathrm{rwt}_\mu(D,k)\Big)t^{\frac12}\tau_7^\vee t^{\frac12}\tau_9^\vee t^{\frac12}\tau_{10}^\vee \\
&\quad\cdot E_{(\mu_4,\mu_2,\mu_3,\mu_5,\mu_7,\mu_6,\mu_8,\mu_9,\mu_{10},\mu_{11},\mu_1-1)} \\
&= t^{\overset{+}{-}5\overset{-}{+}\#\{\mu_i<\mu_1\}}\Big(\prod_{k\in\{2,3,6,11\}} \mathrm{rwt}_\mu(D,k)\Big)t^{\frac12}\tau_7^\vee t^{\frac12}\tau_9^\vee \\
&\quad\cdot E_{(\mu_4,\mu_2,\mu_3,\mu_5,\mu_7,\mu_6,\mu_8,\mu_9,\mu_{10},\mu_1-1,\mu_{11})} \\
&= t^{\overset{+}{-}5\overset{-}{+}\#\{\mu_i<\mu_1\}}\Big(\prod_{k\in\{2,3,6,10,11\}} \mathrm{rwt}_\mu(D,k)\Big)t^{\frac12}\tau_7^\vee \\
&\quad\cdot E_{(\mu_4,\mu_2,\mu_3,\mu_5,\mu_7,\mu_6,\mu_8,\mu_9,\mu_1-1,\mu_{10},\mu_{11})} \\
&= t^{\overset{+}{-}5\overset{-}{+}\#\{\mu_i<\mu_1\}}\Big(\prod_{k\in\{2,3,6,8,10,11\}} \mathrm{rwt}_\mu(D,k)\Big) \\
&\quad\cdot E_{(\mu_4,\mu_2,\mu_3,\mu_5,\mu_7,\mu_6,\mu_9,\mu_8,\mu_1-1,\mu_{10},\mu_{11})}
\end{aligned}
$$