(1) Use row reduction to find the inverse of the following matrix A. **Other techniques will not give credit.** Be sure to show enough steps so that I know that you know what you are doing.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & -1 & 2 & -1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 & 1 & -1 \end{array}\right].$$

So $A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 0 & 1 \\ -2 & 1 & -1 \end{bmatrix}$.

Check: $AA^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ 0 & 0 & 1 \\ -2 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$

(2) Suppose that A is an $n \times n$ matrix such that $A^3 = \mathbf{0}$. Prove that $I + A$ is invertible and $(I + A)^{-1} = I - A + A^2$.

Find a matrix B such that $(I+A)B = I$ (this automatically implies $B(I+A) = I$ by a theorem proved in class). Take $B = I - A + A^2$.

Then $(I+A)B = (I+A)(I - A + A^2) = I - A + A^2 + A - A^2 + A^3 = I + A^3 = I + 0 = I$.

(3) Use the linearity properties to prove that the transformation $T : \mathbb{R}^3 \mapsto \mathbb{R}^2$ defined below is linear.

$$T([x, y, z]^t) = [x + z, y + 2x]^t.$$

We must prove that $T\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right)$

and that $T\left(k\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = kT\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$.

Check: $T\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{bmatrix}\right) = \begin{bmatrix} x_1+x_2+z_1+z_2 \\ y_1+y_2+2(x_1+x_2) \end{bmatrix}$

$= \begin{bmatrix} x_1+z_1+x_2+z_2 \\ y_1+2x_1+y_2+2x_2 \end{bmatrix} = \begin{bmatrix} x_1+z_1 \\ y_1+2x_1 \end{bmatrix} + \begin{bmatrix} x_2+z_2 \\ y_2+2x_2 \end{bmatrix} =$

$T\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right)$.

And $T\left(k\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = T\left(\begin{bmatrix} kx \\ ky \\ kz \end{bmatrix}\right) = \begin{bmatrix} kx+kz \\ ky+2kx \end{bmatrix} = k\begin{bmatrix} x+z \\ y+2x \end{bmatrix} = kT\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$

(4) Let A be a 3×4 matrix having rank 3. Prove that there is a 4×3 matrix B such that $AB = I$.

Let A be a rank 3 matrix of dimension 3×4. Then A is row equivalent to $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, so the rows of A are linearly independent, and there is a 4×4 matrix R such that

$$AR = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$ (R is a product of row operations).

Let $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, so $ARC = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Thus taking $B = RC$, we get a 4×3 matrix such that $AB = ARC = I$.

(5) We use the ordered basis $\mathcal{B} = \{[1,1]^t, [0,2]^t\}$ to define coordinates for $\mathbb{R}^2$.

(a) Find the point matrix $P_{\mathcal{B}}$.

$$P_B = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}.$$

(b) Find the coordinate matrix $C_{\mathcal{B}}$.

$$C_B = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}^{-1}.$$

$$\left[\begin{array}{cc:cc} 1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc:cc} 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{cc:cc} 1 & 0 & 1 & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} \end{array}\right].$$

So $C_B = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$

Check: $\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$

(c) What point in $X \in \mathbb{R}^2$ has coordinate vector $X' = [3,4]^t$?

$X' = C_B X$, so $X = C_B^{-1} X' = P_B X'$.

Hence $X = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \end{bmatrix}.$

(d) Find the coordinate vector X' for the point $X = [5,1]^t$.

$X' = C_B X$, so $X' = \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}.$

(6) Recall that $\mathcal{P}_2$ is the space of all polynomials of the form $f(x) = a + bx + cx^2$ where $a, b, c \in \mathbb{R}$. Compute the matrix M with respect to the standard ordered basis for $\mathcal{P}_2$ for the linear transformation $L : \mathcal{P}_2 \to \mathcal{P}_2$ where

$$L(f(x)) = f'(x) + f(x).$$

$B = \{1, x, x^2\}$, and $f(x) = a + bx + cx^2$.

Hence $L(f(x)) = L(a+bx+cx^2) = (a+bx+cx^2)' + (a+bx+cx^2)$

$= (b+2cx) + (a+bx+cx^2) = (a+b) + (b+2c)x + cx^2$.

Thus $M\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} a+b \\ b+2c \\ c \end{bmatrix}$, so

$M = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ since $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+b \\ b+2c \\ c \end{bmatrix}$.

(7) Prove that if A is an $n \times n$ matrix for which there exists an $n \times n$ matrix B such that $BA = I$ then A is invertible and $B = A^{-1}$.

We first show that A is invertible. Recall that $\text{rank}(BA) \leq \text{rank}(A)$. Hence

$n = \text{rank}(I) = \text{rank}(BA) \leq \text{rank}(A) \leq n$

since A is $n \times n$ and so has maximal rank n. Thus $\text{rank}(A) = n$, so A is nonsingular and therefore invertible. Since A is invertible, there is necessarily (by definition) an $n \times n$ matrix B such that $BA = I$.

(8) Let $T : \mathcal{V} \to \mathcal{W}$ be a linear transformation between two vector spaces such that the nullspace of T is $\{\mathbf{0}\}$. Let $\{X_1, X_2, X_3\}$ be a set of three linearly independent elements in $\mathcal{V}$. Prove that then $\{T(X_1), T(X_2), T(X_3)\}$ is linearly independent.

Consider the equation $a_1T(X_1)+a_2T(X_2)+a_3T(X_3)=0$.
Then $T(a_1X_1+a_2X_2+a_3X_3)=0$, so $a_1X_1+a_2X_2+a_3X_3$ is in the nullspace of T. Hence $a_1X_1+a_2X_2+a_3X_3=0$.
Since $\{X_1, X_2, X_3\}$ is linearly independent, $a_1=a_2=a_3=0$. Hence $\{T(X_1), T(X_2), T(X_3)\}$ is linearly independent.