1. Explain why each of the statements is either true or false. If false, please provide a counterexample.

   (a) A set that contains a linearly independent set is linearly independent.

   (b) The nullspace of a 3×4 matrix can consist only of the zero vector.

   (c) Homogeneous systems of linear equations are always consistent.

   (d) A system of equations with more variables ("unknowns") than equations must have infinitely many solutions.

(a) False. Let $S \subset V$ be a linearly independent set in a vector space V. Let $T = S \cup \{0\}$ be S plus the zero vector. Then T is linearly dependent ($0 \in T$ is a linear combination of vectors in S) but $S \subset T$.

(b) False. Let $A = 0$ be the 0 matrix (all entries of A are 0). Then for any vector X, $AX = 0$, so $\text{Null}(A) = \mathbb{R}^4$ (A is 3×4).

(c) True. The 0 vector $0 \in \mathbb{R}^n$ is always a solution to $AX = 0$ (usually, $AX = B$, but $B = 0$ since the system is homogeneous)

(d) $\begin{cases} x+y+z=0 \\ x+y+z=1 \end{cases}$ is an inconsistent system of equations.

2. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 0 & -1 & 0 & 1 \end{bmatrix}.$$

(a) Write down the system of equations corresponding to A, viewed as an augmented matrix.

(b) Find the reduced row echelon form of A.

(c) State the general solution of this system of equations, and identity the translations vector and spanning vectors, if there are any.

(d) Provide a linearly independent set of vectors which span the column space of A.

(a) $$\begin{cases} w+x+y+z=1 \\ w \qquad\qquad\quad =1 \\ w-x+y-z=1 \\ w \quad -y \qquad =1 \end{cases}$$

(b) $$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 0 & -1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & -1 & 0 \\ 0 & -2 & 0 & -2 & 0 \\ 0 & -1 & -2 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(c) $z = a$ (a free variable, $a \in \mathbb{R}$)

$y = 0$

$x = -z = -a$

$w = 1$

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\text{translation}} + a \underbrace{\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}}_{\text{spanning}}$$

(d) $$\mathrm{Col}(A) = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

3. Consider the matrix

$$A = \begin{bmatrix} 4 & 5 & 3 & 3 \\ 1 & 1 & 1 & 1 \\ 2 & 3 & 1 & 1 \\ 5 & 7 & 3 & 3 \end{bmatrix}.$$

(a) Calculate the rank of the homogeneous system of linear equations $AX = 0$ whose coefficient matrix is A.

(b) Find the nullspace of A. Describe it as the span of a set of linearly independent vectors in $\mathbb{R}^4$.

(a) Put A into echelon form and count the number of nonzero rows.

$$\begin{bmatrix} 4 & 5 & 3 & 3 \\ 1 & 1 & 1 & 1 \\ 2 & 3 & 1 & 1 \\ 5 & 7 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 1 & 1 \\ 4 & 5 & 3 & 3 \\ 5 & 7 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 2 & -2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence $\operatorname{rank}(A) = 2$.

(b) $$\operatorname{Null}(A) = \left\{ \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} \in \mathbb{R}^4 \;\middle|\; \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2r - 2s \\ r + s \\ r \\ s \end{bmatrix} \right\}$$

Since y and z are free variables.

Hence $$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = r \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix},$$ so $\operatorname{Null}(A) = \operatorname{Span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

4. Consider the system of equations

$$\begin{aligned} x + y + z &= a \\ 4x + 3y + 5z &= b \\ 2x + y + 3z &= c. \end{aligned}$$

(a) Find an echelon form of the augmented matrix associated to this system of equations.

(b) Describe the set of vectors $B = [a, b, c]^t \in \mathbb{R}^3$ such that the system of linear equations is consistent.

(c) Choose a value of $B = [a, b, c]^t$ such that the system is consistent and calculate the rank of the resulting system of linear equations.

(a) $\begin{bmatrix} 1 & 1 & 1 & a \\ 4 & 3 & 5 & b \\ 2 & 1 & 3 & c \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & a \\ 0 & -1 & 1 & -4a+b \\ 0 & -1 & 1 & -2a+c \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 1 & 1 & a \\ 0 & -1 & 1 & -4a+b \\ 0 & 0 & 0 & 2a-b+c \end{bmatrix}.$

(b) The system is consistent if and only if $2a-b+c=0$.

(c) Take $B = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$. Then echelon form of A

is $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, so the rank is 2.

5. (a) Show that the set of vectors $[x, y]^t \in \mathbb{R}^2$ such that $x^2 + y^2 = 1$ is not a subspace of $\mathbb{R}^2$.

(b) Let A be an $m \times n$ matrix. Show that the subset of $\mathbb{R}^n$ consisting of those vectors X such that $AX = 0$ is a subspace of $\mathbb{R}^n$.

(c) Let V be the set of all pairs of real numbers (a, b). Given two elements (a, b) and (c, d) of V, define "addition" by the formula

$$(a, b) + (c, d) = (ac, bd),$$

and given a real number k, define "scalar multiplication" by the formula

$$k(a, b) = (ka, kb).$$

Do these operations make V into a vector space? Why or why not?

(a) Let $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \,\middle|\, x^2+y^2=1 \right\}$. Then $0 \notin V$, so V is not a subspace of $\mathbb{R}^2$.

(b) Let $X, Y \in \mathrm{Null}(A)$. Then $AX=0$ and $AY=0$. Hence $A(aX+bY) = aAX + bAY = 0+0 = 0$, so $aX+bY \in \mathrm{Null}(A)$. Thus $\mathrm{Null}(A)$ is a subspace of $\mathbb{R}^n$.

(c) No. One of the axioms is that, for any scalars s and t, $(s+t)X = sX + tX$. Take $X = (a,b)$. Then $(s+t)(a,b) = ((s+t)a, (s+t)b)$, whereas $s(a,b) + t(a,b) = (sa, sb) + (ta, tb) = (sta^2, stb^2)$. So if $s=t=a=b=1$, for instance, $(2,2) \neq (1,1)$.