Representation Theory Lecture 8, 13 August 2015, Univ. of Melbourne ①

A *crystal* is a collection B of paths which is closed under the action of $\tilde{e}_1, \tilde{e}_2, \tilde{f}_1, \tilde{f}_2$.

Let B_1 and B_2 be crystals.

The *tensor product* of B_1 and B_2 is

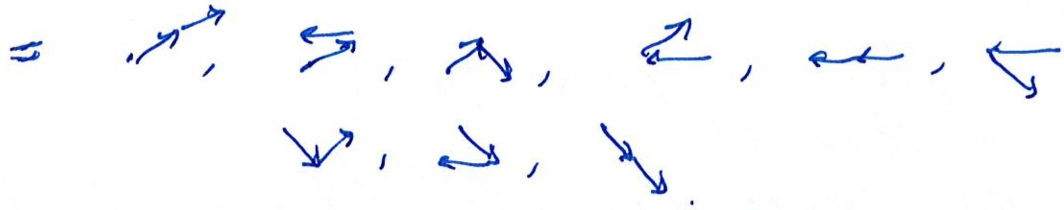
$$B_1 \otimes B_2 = \{ p \otimes q \mid p \in B_1,\ q \in B_2 \}$$

where $p \otimes q$ is the concatenation of p and q.

Example Let $B = \{p_1, p_2, p_3\}$ where

Then

$$B \otimes B = \left\{ \begin{matrix} p_1 \otimes p_1,\ p_1 \otimes p_2,\ p_1 \otimes p_3,\ p_2 \otimes p_1,\ p_2 \otimes p_2,\ p_2 \otimes p_3 \\ p_3 \otimes p_1,\ p_3 \otimes p_2,\ p_3 \otimes p_3 \end{matrix} \right\}$$



Let p be a path, $p : [0,1] \to \mathfrak{h}_{\mathbb{R}}^*$.

Let t_{min} be such that $\langle p(t_{min}), \alpha_i^\vee \rangle$ is minimum

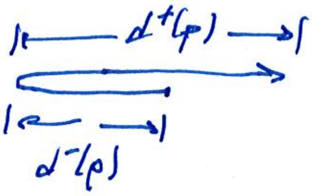
$\langle p(t), \alpha_i^\vee \rangle$ is the "distance from $p(t)$ to $\mathfrak{h}^{\alpha_i}$"

i.e. $p(t_{min})$ is the ~~leftmost~~ most negative point of p.

Define

$$d_i^-(p) = |\langle p(0) - p(t_{min}), \alpha_i^\vee \rangle|$$

$$d_i^+(p) = |\langle p(1) - p(t_{min}), \alpha_i^\vee \rangle|$$



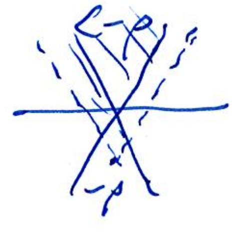
Proposition Let p and q be paths. Then

$$\tilde{f}_i(p \otimes q) = \begin{cases} \tilde{f}_i p \otimes q, & \text{if } d_i^+(p) > d_i^-(q) \\ p \otimes \tilde{f}_i q, & \text{if } d_i^+(p) \le d_i^-(q) \end{cases}$$

$$\tilde{e}_i(p \otimes q) = \begin{cases} \tilde{e}_i p \otimes q, & \text{if } d_i^+(p) \ge d_i^-(q) \\ p \otimes \tilde{e}_i q, & \text{if } d_i^+(p) < d_i^-(q). \end{cases}$$

Highest weight paths

A highest weight path is a path p such that

$p \subseteq C - p$ where  so that 

A crystal B is irreducible if its crystal graph is connected.

Proposition Let B be a crystal.

B is irreducible if and only if

B contains a unique highest weight path.

<u>Proposition</u> Let B_1 and B_2 be irreducible crystals. Let p be the hw path of B_1 and let q be the hw path of B_2.

Then $B_1 \simeq B_2$ if and only if $p(1) = q(1)$

(the endpoint of p is the same as the endpoint of q).

<u>Theorem</u> The irreducible crystals are indexed by elements of $(C-\rho) \cap \mathfrak{h}^*_{\mathbb{Z}}$, i.e. the function

$$\left\{\begin{matrix}\text{irreducible}\\ \text{crystals}\end{matrix}\right\} \longleftrightarrow (C-\rho) \cap \mathfrak{h}^*_{\mathbb{Z}}$$

$$B(p_\lambda) \longleftarrow\!\!\shortmid \lambda$$

is a bijection.

p_λ is the straight line path to λ

$B(p_\lambda)$ is the crystal generated by the action of $\tilde{f}_1$ and $\tilde{f}_2$ on p_λ..

The set of <u>dominant integral weights</u> is

$$(\mathfrak{h}^*_{\mathbb{Z}})^+ = (C-\rho) \cap \mathfrak{h}^*_{\mathbb{Z}}.$$

<u>Characters</u>

Let B be a crystal. The <u>character</u> of B is

$$\mathrm{char}(B) = \sum_{p \in B} X^{p(1)}.$$

The character of $\mathcal{B}$ is an element of the ring

$$\mathbb{Z}[\mathfrak{h}_{\mathbb{Z}}^*] = \mathbb{Z}\text{-span}\{X^\mu \mid \mu \in \mathfrak{h}_{\mathbb{Z}}^*\}$$

with
$$X^\mu X^\nu = X^{\mu+\nu}$$

Note that

$$\mathrm{char}: \{\text{crystals}\} \longrightarrow \mathbb{Z}[\mathfrak{h}_{\mathbb{Z}}^*]$$

is such that

$$\mathrm{char}(\mathcal{B}_1 \sqcup \mathcal{B}_2) = \mathrm{char}(\mathcal{B}_1) + \mathrm{char}(\mathcal{B}_2)$$

$$\mathrm{char}(\mathcal{B}_1 \otimes \mathcal{B}_2) = \mathrm{char}(\mathcal{B}_1)\, \mathrm{char}(\mathcal{B}_2).$$

The <u>Weyl characters</u> are

$$s_\lambda = \mathrm{char}(\mathcal{B}(p_\lambda)) \qquad \text{for } \lambda \in (\mathfrak{h}_{\mathbb{Z}}^*)^+$$

<u>Reference</u> A. Ram, Alcove walks, Hecke algebras, spherical functions, crystals and column strict tableaux, arXiv:math.RT/0601343.

See also the book of

J. Hong and S.J. Kang, Introduction to quantum groups and crystal bases, Amer. Math Soc. Grad. Studies in Math. vol. 42, 2002.